DOI: 10.24996/ijs.2024.65.10.28

ISSN: 0067-2904

The Degree-Topology of CertainTypes of Graphs and PT_0- Space

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Received: 5/5/2023 Accepted: 27/8/2023 Published: 30/10/2024

Abstract
This work aimed to study the degree topology of certain types of graphs namely, the wheel, helm, gear, pan, tadpole, and windmill graphs, respectively. In addition, a new separation axiom known as PT_0-space that joins the topological space and paths in the graph is introduced. Also, the relation between PT_0-space and the degree-topology of the connected and disconnected graph was stydied. Moreover, many examples were studied in new separation space.

Keywords: Degree-topological Space, PT_0-*space*.

توبولوجي الدرجة لأنواع معينة من الرسوم البيانية و فضاء الفصل $-G_1$

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الخلاصة
في هذا العمل تمت دراسة توبولوجي الدرجة لأنواع معينة من الرسوم البيانية تحديدا الرسوم البيانية للعجلة, الهلم, العتاد, مِقلاة, الشرغوف والطاحونة على التوالي بالإضافة إلى تم تقديم بديهية فصل جديدة تعرف بفضاء $-PT_0$ تعتمد على الربط بين الفضاء التبولوجي و المسارات في الرسم البياني. ايضا, تمت دراسة العلاقة بين فضاء$-PT_0$ وتبولوجي الدرجة للرسم البياني المتصل والمنفصل. علاوة على ذلك, تمت دراسة العديد من الأمثلة لتحقيق بديهية الفصل الجديدة.

1. Introduction

The field of graph theory is broad and diverse. It is a topological space, a combinatorial object, and a relational structure. Graphs can abstractly represent many concepts, making them useful in real-world applications [1]. Certain graphs created topological spaces by various methods. In 1964, Ahlborn defined a unique topological space on a digraph $G(V, E)$ where any set A in V is publicly provided there is no edge from $(V - A)$ to A [2]. In 2013, Hamza and Faisel built a topology on the set of edges of an undirected graph. They created symmetric topological spaces and investigated the property of compactness [3]. Jafarian et al.

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established a sub-basis family for a graphic topology as a collection of all vertices bordering the vertex v and their properties [4]. Moreover, H. Sari and A. Kopuzlu [5] studied topologies generated by simple undirected graphs without isolated vertices and their properties. Also, necessary and sufficient condition for continuity and openness of functions defined from one graph to another are given by using the topologies generated by these graphs.

A. Hassan and Z. Abed investigated new topologies that were created by simple undirected graphs that did not have any isolated vertices and they applied this topology to some main subjects in biomathematics [6]. The authors of [7] presented two types of relations on the edges set that generate topological spaces, they discussed some properties of this topology, also they study the method of returning from the topological space to the graphs through using relationships. Hassan and Jafar [8] introduced a family of sub-basis that produced a new topology that contains all non-end vertices of the edge E on the vertex set V of each simple graph G. Al'Dzhabri et al. introduced DG-topological space with DG-open set and the digraph $G = (V, E)$ as T_{DG} [9].

Hameed and Kadhem [10] introduced a new topology on a graph, namely the degree-topology that is defined by the degree of the vertices of the graphs. Moreover, they initiated a new property, namely set-T_0space. Then, they introduced two concepts that related to the degree-topology in [11], set-texture space and degree-texture spaces and they investigated the relationship between them. In 2023, Z. N. Jwair and M. A. Abdlhusein constructed the topological graph from discrete topology and studied many properties of a new graph namely, simple, undirected, and connected graph. Also, they studied the radius and diameter for the new graph [12].

In this paper, we are going to find the degree-topology of graphs such as wheels, helms, gears, pans, tadpoles, and windmills. In addition, we introduce PT_0-space which is a separation axiom that connects degree-topological space with paths in the graph. Moreover, there are many examples in this work to discuss various concepts that are mentioned. We will refer to a path between two vertices v_i and v_j by $P_{v_iv_j}$ and its length denote by $\left|P_{v_iv_j}\right|$.

2. Preliminaries

This section includes several basic definitions and theorems on a degree-topology that are essential in the paper are reviewed.

Definition 2.1 [10] Let $G(V, E)$ be a simple graph and K be the max degree of all vertices in G. Then, the topology defines on the vertex set V and generated by a basis B is called a degree-topology, and it is denoted by T_{deg}, where $B_{deg} = \{A_i: i = 0, \dots, K\}$, A_i is the set of all vertices that have a degree i, and K is the greatest degree of all vertices in G.

Theorem 2.2 [10] A degree-topology that is generated by a complete graph K_n with n vertices is an indiscrete topology.

Theorem 2.3 [10] A degree-topology which is generated by the cycle graph C_n with n vertices is an indiscrete topology.

Theorem 2.4 [10] Every degree-topology is a set-T_0 space.

Example 2.5 [10] Let $P_4(V, E)$ be a path graph of order four. Then, T_{deg} (P_4) is a degree-texture of V with the degree-relation. Also, it implies the set-texture of V.
Theorem 2.6 [10] A degree-topology that is generated by a complete bipartite graph $K_{n,m}$ with $n = m$ is an indiscrete topology.
Theorem 2.7 [10] A degree-topology that is generated by a complete bipartite graph $K_{n,m}$ with $n \neq m$ is quasi-discrete.
Definition 2.8 [13] The pan graph $N_n(V, E)$ is the graph obtained by joining a cycle graph $C_n(V, E)$ to a vertex by an edge. Therefore, the pan graph is isomorphic with the tadpole graph, $T_{1,m}(V, E)$.
Definition 2.9 [13] The wheel graph $W_n(V, E)$ with n vertices is obtained from the cycle graph C_{n-1} by joining each vertex to a new vertex v_n.

Definition 2.10 [14] The gear graph $G_n(V, E)$ is a wheel graph $W_n(V, E)$ with an additional vertex inserted between each adjacent vertices of outer cycle graph.

Definition 2.11 [15] The helm graph $H_n(V, E)$ is the graph obtained from an wheel graph $W_n(V, E)$ by joning each vertex of the cycle with a pendant vertex by an edge.

Definition 2.12 [16] The tadpole graph, $T_{n,m}(V, E)$ is the graph created by by joining the cycle graph $C_m(V, E)$ and a path graph $P_n(V, E)$ with an edge from any vertex of C_m to a pendent vertex of P_n, where m and n is the order of the cycle graph C_m and path graph P_n, respectively.

Definition 2.13 [17] The windmill graph $W_n^m(V, E)$ is the graph obtained by taking m copies of the complete graph $K_n(V, E)$ with a vertex in common. where n is the order of complete graph K_n.

3. Degree-topology of certain types of graphs

In this section we give some examples to investigate the degree of topology of certain types of graphs as follows:
Example 3.1 The degree-topology generated by the wheel graph W_n with $n = 4$ is an indiscrete topology and if $n > 4$ is a quasi discrete-topology.

Solution: Assume that $W_n(V, E)$ is a wheel graph of order n, as in Figure 1 with the vertex set $V = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$, where v_n is the universal vertex and $\{v_1, v_2, .., v_{n-1}\}$ is the vertices of the cycle graph C_{n-1}. By definition of the wheel graph, we have two cases:

- If $n = 4$, the vertex v_4 connected with all vertices of C_3 then the degree of v_4 is three. Since every vertex of C_3 have the degree two and each vertex joins withthe universal vertex v_4 by an edge then, the degree at each vertex of C_3 is three. We have $A_0 = \emptyset$, $A_3 = \{v_1, v_2, v_3, v_4\}$, so that the basis for T_{deg} is $\{\emptyset, V\}$ and by taking all unions the degree-topology generated by W_4 is an indiscrete topology.
- If $n > 4$, the vertex v_n connected with all vertices of C_{n-1} then the degree of v_n is $n - 1$. Since every vertex of C_{n-1} have the degree two and each vertex joins with the universal vertex v_n by an edge then, the degree at each vertex of C_{n-1} is three. We have $A_0 = \emptyset$, $A_3 = \{v_1, v_2, .., v_{n-1}\}$ and $A_{n-1} = \{v_n\}$, so that the basis for T_{deg} is $\{\emptyset, \{v_n\}, \{v_1, v_2, .., v_{n-1}\}\}$ and by taking all unions the degree-topology generated by the wheel graph is$\{\{V, \emptyset, \{v_n\}, \{v_i\}_{i=1}^{n-1}\}$. Thus, the degree-topology generated by the Wheel graph W_n with $n = 4$ is an indiscrete topology and if $n > 4$ is a quasi discrete topology. ■

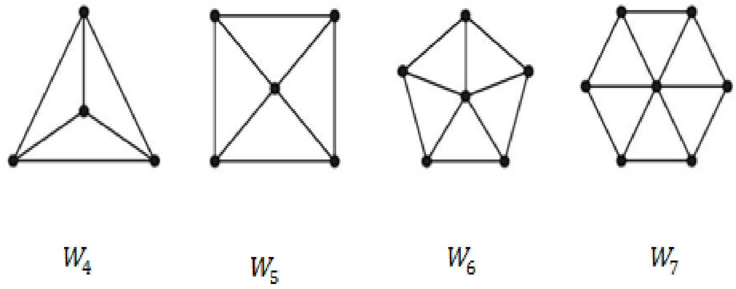

Figure 1: The wheel graph $W_4, W_5, W_6$and W_7

Example 3.2 The degree-topology generated by the helm graph H_5 is an quasi discrete-topology.

Solution: Assume that $H_5(V,E)$ is a helm graph of order nine and $W_5(V,E)$ be a wheel graph in H_5 as in Figure 2. By definition, the wheel graph contains a cycle graph C_4 and the vertex v_5 connected with all vertices of C_4 such that $V(C_4) = \{v_1, v_2, v_3, v_4\}$. By definition of the helm graph, there is $\{u_1, u_2, u_3, u_4\}$ a pendent vertex that joins with cycle vertices. We will be denoted by the degree of vertex v by $\rho(v)$, we have two cases for the degree of helm vertices:

- $\rho(u_i)$= 1 with $i = 1,2,3,4$ for it is a pendent vertex.
- $\rho(v_j)$= 4 with $j = 1,2,3,4,5$ for each vertex in C_4 join with the vertex v_4 and one of the pendent vertices.

Thus, A_0=$\emptyset$, A_1=$\{u_1, u_2, u_3, u_4\}$ and A_4=$\{v_1, v_2, v_3, v_4, v_5\}$. Hence, the basis for T_{deg} is {$\emptyset$, $\{u_1$,u_2, $u_3, u_4\}$,$\{v_1$,v_2, v_3,v_4, $v_5\}$}and by taking all unions the degree-topology generated by the helm graph is $\{V, \emptyset, \{v_i\}_{i=1}^{5}, \{u_i\}_{i=1}^{4}\}$.■

Example 3.3 The degree-topology generated by the helm graph H_n with $n = 4$ or $n > 5$ is $\{V, \emptyset, \{v_n\}, \{v_i\}_{i=1}^{n-1}, \{u_i\}_{i=1}^{n-1}, \{v_i\}_{i=1}^{n}, \{v_n, u_i\}_{i=1}^{n-1}\}, \{u_i, v_i\}_{i=1}^{n-1}\}\}$, where v_n be the vertex that joins with $n-1$ vertices of the cycle graph C_{n-1}, the vertices $\{u_i\}_{i=1}^{n-1}$ be pendent vertices, and $\{v_i\}_{i=1}^{n-1}$ be a vertex of the cycle graph C_{n-1}.

Solution: Assume that $H_n(V,E)$ is a Helm graph of order $2n-1$ and $W_n(V,E)$ be a wheel graph in helm graph with $n = 4$ or $n > 5$ as in Figure 2. By definition, the wheel graph contains a cycle graph C_{n-1} and the vertex v_n connected with all vertices of C_{n-1}such that $V(C_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$. By definition of the helm graph, there is $\{u_1, u_2, .., u_{n-1}\}$ pendent vertices that join with cycle vertices. We will be denoted by the degree of vertex v by $\rho(v)$, we have three cases for the degree of helm vertices:

- $\rho(v_n)$= $n-1$ for join with $n-1$ vertices.
- $\rho(u_i)$= 1 with $i = 1,2, \ldots, n-1$ for it is a pendent vertex.
- $\rho(v_j)$= 4 with $j = 1,2, \ldots, n-1$ for each vertex in C_{n-1}join with the vertex v_n and pendant vertex.

Thus, $A_0 = \emptyset$, $A_1 = \{u_1, u_2, .., u_{n-1}\}$, $A_4 = \{v_1, v_2, .., v_{n-1}\}$ and $A_{n-1} = \{v_n\}$. Hence, the basis for T_{deg} is $\{\emptyset, \{v_n\}, \{v_i\}_{i=1}^{n-1}, \{u_i\}_{i=1}^{n-1}\}$ and by taking all unions the degree-topology generated by the helm graph is $\{V, \emptyset, \{v_n\}, \{v_i\}_{i=1}^{n-1}, \{u_i\}_{i=1}^{n-1}, \{v_i\}_{i=1}^{n}, \{v_n, u_i\}_{i=1}^{n-1}\}, \{u_i, v_i\}_{i=1}^{n-1}\}$.
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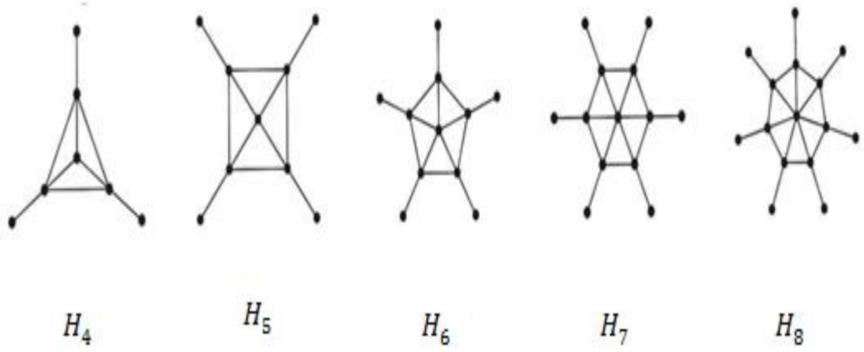

Figure 2: The helm graph $H_4, H_5, H_6, H_7$and H_8

Example 3.4 The degree-topology generated by a gear graph G_4 is an quasi discretetopology.
Solution: Assume that $G_4(V, E)$ is a gear graph of order 7 as in Figure 3 and $W_4(V, E)$ be a wheel graph in gear graph. By definition, the wheel graph contains a cycle graph C_3 and the vertex v_4 connected with all vertices of C_3such that $V(C_3) = \{v_1, v_2, v_3\}$.Then the degree of $\{v_1, v_2, v_3, v_4\}$ is three. By definition of the gear graph, there is $\{s_1, s_2, s_3\}$ vertices that are added between each adjacent pair of vertices in the outer of the cycle, so that the degree of $\{s_1, s_2, s_3\}$ is two. We have, $A_0 = \emptyset$, $A_2 = \{s_1, s_2, s_3\}$, and $A_3 = \{v_1, v_2, v_3, v_4\}$ Hence, the basis for T_{deg} is $\{\emptyset, \{s_1, s_2, s_3\}, \{v_1, v_2, v_3, v_4\}\}$ and by taking all unions the degree-topology generated by the gear graph is $\{V, \emptyset, \{v_i\}_{i=1}^{4}, \{s_i\}_{i=1}^{3}\}$.■

Example 3.5 The degree-topology generated by the gear graph G_n with $n > 4$ is $\{V, \emptyset, \{v_n\}, \{v_i\}_{i=1}^{n-1}, \{s_i\}_{i=1}^{n-1}, \{v_i\}_{i=1}^{n}, \{v_n, s_i\}_{i=1}^{n-1}, \{v_i, s_i\}_{i=1}^{n-1}\}$, where v_n be the vertex that joins with $n - 1$ vertices of the cycle graph C_{n-1} that is $\{v_1, v_2, .., v_{n-1}\}$, and $\{s_1, s_2, .., s_{n-1}\}$ be vertices that are added between each adjacent pair of vertices in the outer of the cycle.

Solution: Assume that $G_n(V, E)$ is a gear graph of order $2n - 1$and $W_n(V, E)$ be a wheel graph in Gear graph with $n \geq 4$ see a Figure 3. Since the wheel graph W_n contains a cycle graph C_{n-1} we have the degree of a vertex v_n is $n - 1$ and each vertex in $V(C_{n-1})$ has a degree three. By definition of the gear graph, there is $\{s_1, s_2, .., s_{n-1}\}$ vertices that are added between each adjacent pair of vertices in the outer of the cycle so the degree of $\{s_1, s_2, .., s_{n-1}\}$ is two. Thus, $A_0 = \emptyset$, $A_2 = \{s_1, s_2, .., s_{n-1}\}$, $A_3 = \{v_1, v_2, .., v_{n-1}\}$ and $A_{n-1} = \{v_n\}$. Hence, the basis for T_{deg} is $\{\emptyset, \{v_n\}, \{s_1, s_2, .., s_{n-1}\}, \{v_1, v_2, .., v_{n-1}\}\}$ and by taking all unions the degree-topology generated by the gear graph is$\{V, \emptyset, \{v_n\}, \{v_i\}_{i=1}^{n-1}, \{s_i\}_{i=1}^{n-1}, \{v_i\}_{i=1}^{n}, \{v_n, s_i\}_{i=1}^{n-1}, \{v_i, s_i\}_{i=1}^{n-1}\}$. ■

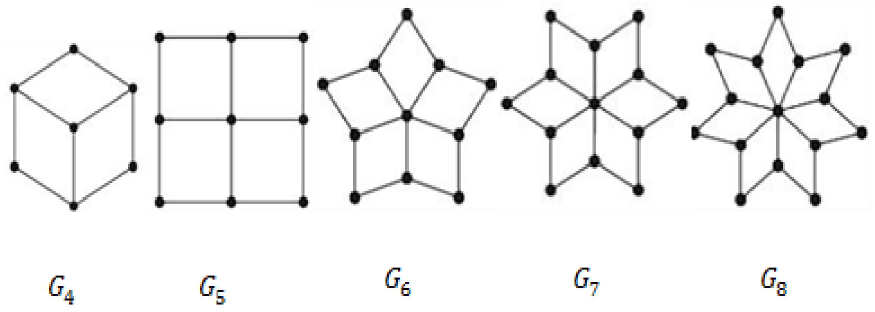

Figure 3: The gear graph $G_4, G_5, G_6, G_7$and G_8

Example 3.6 The degree-topology generated by the pan graph N_n with $n \geq 3$ is $\{V, \emptyset, \{v_k\}, \{v_{n+1}\}, \{v_i\}_{i=1,i\neq k}^{n}, \{v_k, v_{n+1}\}, \{v_i\}_{i=1}^{n}, \{v_i\}_{i=1,i\neq k}^{n+1}\}$,where v_k be a vertex in the cycle graph C_n that join a pendent vertex v_{n+1}, and $\{v_i\}_{i=1}^{n}$ be a vertex of the cycle graph C_n in a pan graph, for any $i \neq k$.

Solution: Let $N_n(V, E)$ is a pan graph of order $n + 1$ as in Figure 4 and $C_n(V, E)$ be a cycle graph in a pan graph N_n with $n \geq 3$ where $V(C_n) = \{v_1, v_2, \dots, v_n\}$. By definition of the pan graph, there is a vertex v_{n+1} connected with one vertex of C_n say v_k. We will be denoted by the degree of vertex v by $\rho(v)$, we have three cases for the degree of pan vertices:

- $\rho(v_i)$= 2with $i = 1,2, \dots, n,$ and $i \neq k$ for each vertex in C_n has a degree two.
- $\rho(v_{n+1})$= 1 for it is a pendent vertex.
- $\rho(v_k)$= 3 for it to join with the vertex v_{n+1}.

Thus, $A_0 = \emptyset$, $A_1 = \{v_{n+1}\}$, $A_3 = \{v_k\}$and $A_2 = \{v_i\}_{i=1}^{n}$, with $i \neq k$. Hence, the basis for T_{deg} is $\{\emptyset, \{v_k\}, \{v_{n+1}\}, \{v_i\}_{i=1}^{n}\}$ and by taking all unions the degree-topology generated by the pan graph is$\{V, \emptyset, \{v_k\}, \{v_{n+1}\}, \{v_i\}_{i=1,i\neq k}^{n}, \{v_k, v_{n+1}\}, \{v_i\}_{i=1}^{n}, \{v_i\}_{i=1,i\neq k}^{n+1}\}$.■

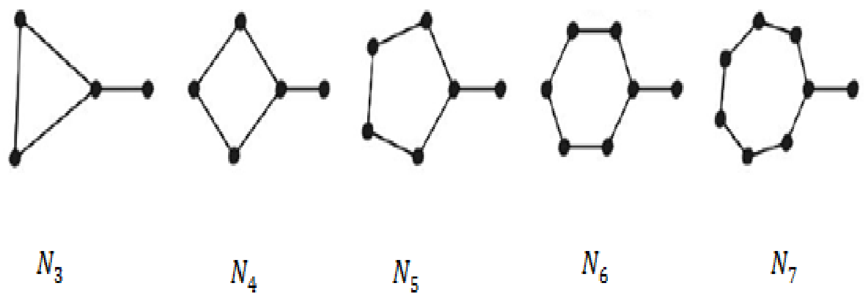

Figure 4: The pan graph N_3, N_4, N_5, N_6 and N_7

Example 3.7 The degree-topology generated by the tadpole graph $T_{(n,m)}$ is $\{V, \emptyset, \{w_i\}_{i=1}^{m+n-2}$, $\{v_k\}, \{u_n\}, \{v_k, u_n\}, \{v_k, w_i\}_{i=1}^{m+n-2}, \{u_n, w_i\}_{i=1}^{m+n-2}\}$ with m and n being the order of the cycle graph C_m and path graph P_n, respectively. Where u_n be a pendent vertex, v_k be a vertex join with u_1 which is another pendent vertex for the path graph P_n in $T_{(n,m)}$ and $\{w_i\}_{i=1}^{m+n-2}$ be the other vertices in $T_{(n,m)}$.

Solution: Let $T_{(n,m)}(V, E)$ be a tadpole graph of order $n + m$ where m and n be the order of cycle graph $C_m(V, E)$ and path graph $P_n(V, E)$, respectively. Assume that $V(C_m) = \{v_1, v_2, \dots, v_m\}$ and $V(P_n) = \{u_1, u_2, \dots, u_n\}$ where u_1 is the first vertex and u_n is the last vertex. By definition of a tadpole graph, the cycle graph join with a path graph by an edge in

$T_{(n,m)}$, suppose, v_k be one vertex of C_m connected with u_1 by an edge. We have three cases for the degree of tadpole vertices:

- The vertex v_k has a degree three.
- The vertex u_n has a degree one.
- Other vertices of a tadpole that is $m + n - 2$ vertices have a degree two labeled those vertices by $\{w_1, w_2, \dots, w_{m+n-2}\}$.

Thus, $A_0 = \emptyset$, $A_1 = \{u_n\}$, $A_3 = \{v_k\}$ and $A_2 = \{w_i\}_{i=1}^{m+n-2}$. Hence, the basis for T_{deg} is $\{\emptyset, \{u_n\}, \{v_k\}, \{w_i\}_{i=1}^{m+n-2}\}$ and by taking all unions the degree-topology generated by the tadpole graph is $\{V, \emptyset, \{w_i\}_{i=1}^{m+n-2}, \{v_k\}, \{u_n\}, \{v_k, u_n\}, \{v_k, w_i\}_{i=1}^{m+n-2}, \{u_n, w_i\}_{i=1}^{m+n-2}\}$. See the Figure 5. ■

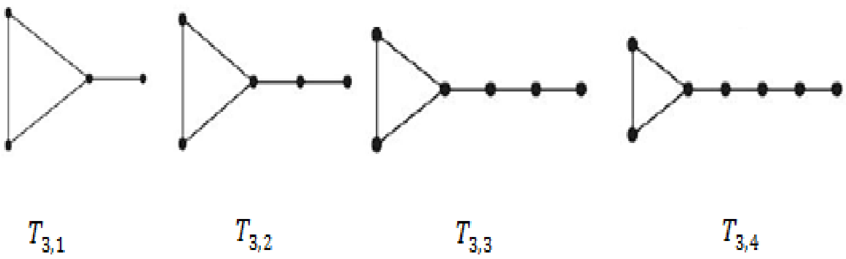

Figure 5: The tadpole graph $T_{3,1}, T_{3,2}, T_{3,3}$ and $T_{3,4}$

Example 3.8 The degree-topology generated by the windmill graph W_n^m is $\{V, \emptyset, \{v\}, \bigcup_{i=1}^{m}\{v_1^i, \dots v_{n-1}^i\}\}$ where v be a vertex that common in all copies of a complete graph K_n of order n in the windmill graph and $\bigcup_{i=1}^{m}\{v_1^i, \dots, v_{n-1}^i\}$ is another vertex of the m copies of the complete graph K_n.

Solution: Let $W_n^m(V, E)$ be a windmill graph of order $m(n-1) + 1$where n is the order of a complete graph $K_n(V, E)$ and m be the number of copies K_n, see a Figure 6. Assume that $V(K_n)$=$\{v_1, v_2, \dots, v_n\}$ and v be a vertex that common in all copies of K_n. We have two cases for the degree of a windmilll vertex:

- The vertex v has a degree $m(n-1)$.
- Another vertex of the windmill that is $m(n-1)$vertices has a degree $n-1$ labeled those vertices by $\bigcup_{i=1}^{m}\{v_1^i, \dots, v_{n-1}^i\}$.

Thus, $A_0 = \emptyset$, $A_{m(n-1)} = \{v_j\}$, and $A_{n-1} = \bigcup_{i=1}^{m}\{v_1^i, \dots, v_{n-1}^i\}$. Hence, the basis for T_{deg} is $\{\emptyset, \{v\}, \bigcup_{i=1}^{m}\{v_1^i, \dots v_{n-1}^i\}\}$ and by taking all unions the degree-topology generated by the windmill graph is $\{V, \emptyset, \{v\}, \bigcup_{i=1}^{m}\{v_1^i, \dots v_{n-1}^i\}\}$. ■

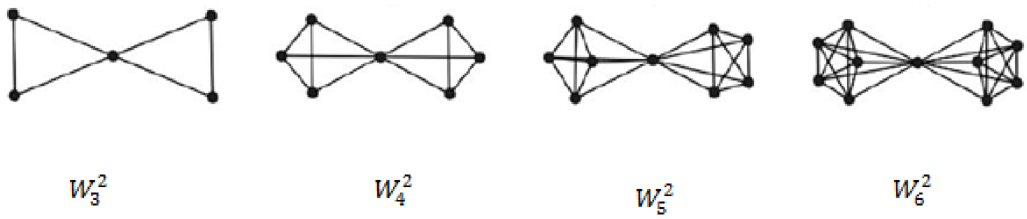

Figure 6: The windmill graph W_3^2, W_4^2, W_5^2 and W_6^2

4. PT_0-space

This section aimed to introduce a new separation axiom.

Definition 4.1 Let $G(V,E)$ be a simple graph and let T_{deg} be a degree-topology on G. Then, $T_{deg}(G)$ is said to satisfied a PT_0- separation axiom if for any distinct vertices v_1 and v_2 in $V(G)$, there exists a path in G of the length greatest or equal to two, or there exists an open set W in $T_{\text{deg}}(G)$ such that W includes just one of these vertices. For there the $(T_{deg}(G), G)$ is called a PT_0- space.

Example 4.2 Let $G(V,E)$ be a graph as in Figure 7. Where V={v_1,v_2,v_3,v_4, v_5,v_6, $v_{7,}$$v_8$,v_9,v_{10}}, then $T_{\text{deg}}(G)$={$\emptyset$, V,{v_1, v_2, v_{10}},{v_4, v_7,v_6,v_9},{v_5, v_8, v_3},{v_1, v_2,v_{10} ,v_4,v_7,v_6,v_9} ,{v_1, v_2, v_{10}, v_5, v_8, v_3},{v_4,v_7, v_6, v_9, v_5, v_8, v_3}} is a PT_0-space due to for the pairs v_1, v_2, there is a path v_1,v_3,v_2 of length 2, for each pairs of v_1, v_{10} and v_2, v_{10} there are paths of length greater than 2, while for v_i $(i = 1,2,10)$ and v_j $(j = 3,4,5,6,7,8,9)$ there is open set $\{v_1, v_2, v_{10}\}$ which contains v_i but not v_j, so on for the other cases that are related in the same way. Therefore, $T_{deg}(G)$ the PT_0-separating axiom.

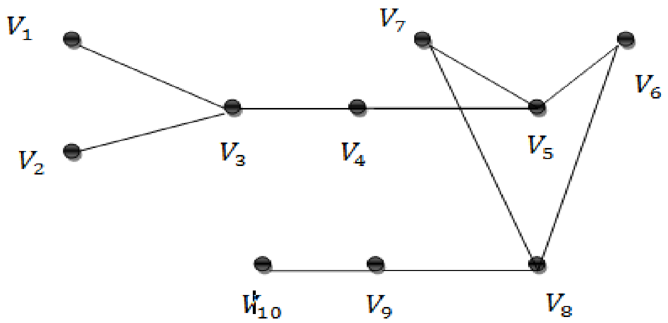

Figure 7: The graph $G(V,E)$

Example 4.3 Let $P_5(V,E)$ be a path graph, where $V = \{v_1, v_2, v_3, v_4, v_5\}$, then $T_{\text{deg}}(P_5)$= $\{\emptyset, V, \{v_1, v_5\}, \{v_2, v_3, v_4\}\}$. Then, $T_{deg}(P_5)$ is not a PT_0-space for v_3, and v_2 in $V(P_5)$. The only v_2v_3-path in G of the length one. Also, there is not an open set in $T_{deg}(P_5)$ that contains just one of them.

Remark 4.4 Every degree-topology generated by the path graph $P_n(V,E)$ of order n, does not satisfy the PT_0-separation axiom with $n \neq 3$, for any vertex in $P_n(V,E)$ is of degree one or of degree two. So, the degree-topology $T_{deg}(P_n) = \{\emptyset, V, \{v_1, v_n\}, \{v_2, v_3, \dots, v_{n-1}\}\}$, and so v_3 and v_2 are adjacent. This means, there is only one path between them. Also, there is no open set containing just one of these vertices.

Example 4.5 Let $P_3(V,E)$ be a path graph of order 3, where $V = \{v_1, v_2, v_3\}$. Then, $T_{\text{deg}}(P_3) = \{\emptyset, V, \{v_1, v_3\}, \{v_2\}\}$ is a PT_0-space for vertices v_1, and v_2 in $V(P_3)$. Then there exists an open set $\{v_2\}$ in $T_{deg}(P_3)$ such that $v_2 \in \{v_2\}$, and $v_1 \notin \{v_2\}$. Hence there is a v_1v_3-path in P_3 of the length greatest or equal to two. Consequently, there exists an open set $\{v_2\}$ in $T_{deg}(P_3)$ such that $v_2 \in \{v_2\}$ and $v_3 \notin \{v_2\}$. See Figure 8.

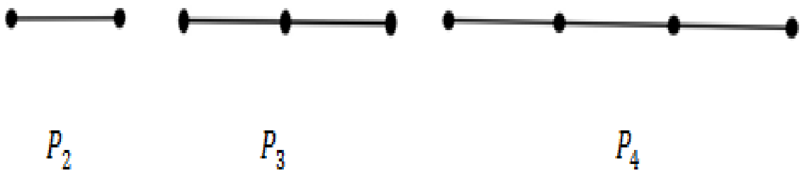

Figure 8: The path graph P_2, P_3 and P_4

Example 4.6 Let $C_n(V,E)$ be a cycle graph of order 4, where $V = \{v_1, v_2, v_3, v_4\}$. Then, $T_{\deg}(C_n) = \{\emptyset, V\}$ is PT_0-space for vertices v_1 and v_2 in $V(C_n)$, there is $P_{v_1v_2}$ with $|P_{v_1v_2}| \geq 2$. Similarly, for the other vertices. Thus, $T_{deg}(C_n)$ is a PT_0-space.

Theorem 4.7: The degree-topology generated by cycle graph C_n with n vertices is a PT_0-space.
Proof: Let $C_n(V,E)$ be the cycle graph of order n, where $V = \{v_1, v_2, v_3, \dots, v_n\}$. By using mathematical induction we can prove that the degree-topology $T_{deg}(C_n)$ is a PT_0-space, as follows:

- Base step: If $n = 3$, then $T_{deg}(C_3) = \{\emptyset, V\}$ is a PT_0-space for vertices v_1 and v_2 in $V(C_3)$, there exists $P_{v_1v_2}$ with $|P_{v_1v_2}| \geq 2$. For vertices v_1 and v_3 in $V(C_3)$, there is $P_{v_1v_3}$ with $|P_{v_1v_3}| \geq 2$. Finally, for vertices v_2 and v_3 in $V(C_3)$, there exists $P_{v_2v_3}$ with $|P_{v_2v_3}| \geq 2$. Hence, $T_{deg}(C_3)$ is a PT_0-space.
- Inductive hypothesis: Suppose that for all $n \geq 3$, the degree-topology generated by the cycle graph C_n with n vertices is a PT_0-space.

Purpose Inductive step: For $k = n + 1$, where $n \in \mathbb{Z}^+$. We must prove that the degree-topology generated by the cycle graph $C_k(V,E)$ with k vertices is satisfy a PT_0-separation axiom. The cycle graph $C_k(V,E)$, with k vertices is shown by the cycle graph C_n with n vertices that is added the vertex v_{n+1} between the adjacent vertices v_1 and v_n. This means the vertex v_{n+1} is divides the path v_1,v_n into two paths which are v_1,v_{n+1} *and* v_n,v_{n+1} as it shown in Figure 9. For vertices v_1 and v_{n+1} in $V(C_k)$, there exists a v_1v_{n+1}-path is $v_1, v_2, v_3, \dots, v_{n-1}, v_n, v_{n+1}$ in C_k of the length greatest or equal to two. Also, for vertices v_n and v_{n+1} in $V(C_k)$, there exists a v_nv_{n+1}- path is $v_{n+1}, v_1, v_2, v_3, \dots, v_{n-1}, v_n$in C_k of the length greatest or equal to two. From the inductive hypothesis, there exists a v_iv_{n+1}- path in C_k of the length greatest or equal to two, for $i = 2,3, \dots, n-1$. Consequently, from the three steps, we get that $T_{deg}(C_n)$ satisfies the PT_0-separation axiom.■

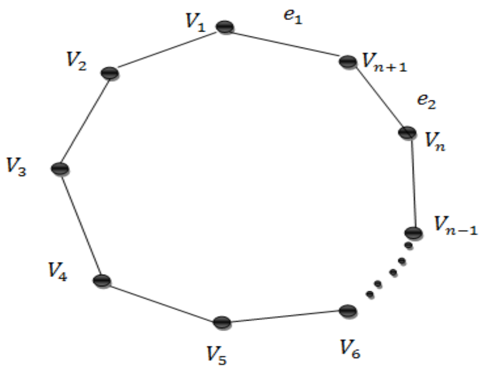

Figure 9: The cycle graph

Theorem 4.8 The degree-topology which is generated by complete graph K_n with n vertices satisfies the PT_0-separation axiom.

Proof: Assume that $K_n(V,E)$ is a complete graph with the vertex set $V(K_n) = \{v_1, v_2, \dots, v_n\}$, see Figure 10. Let v_i and v_j be two distinct vertices in $V(K_n)$, where $i, j = 1,2, \dots, n$.
By definition of a complete graph, any two vertices in K_n are adjacent. Then, v_k is adjacent with v_i and v_j, where $k = 1,2, \dots, n$ and $v_i \neq v_k \neq v_j$. Thus, for any distinct vertices v_i, and v_j there exist $v_i v_j$- path of length equal to two that is v_i, v_k, v_j. Thus, the degree-topology generated by the complete graph satisfies the PT_0-separation axiom.■

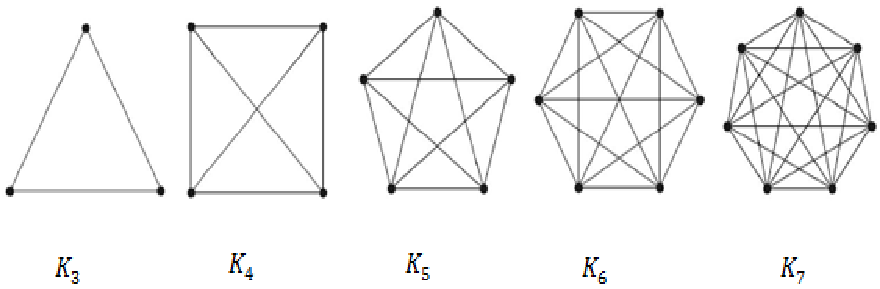

Figure 10: The complete graph K_3, K_4, K_5, K_6 and K_7

Theorem 4.9 Every degree-topology which is satisfied T_0 space is a PT_0-space.

Proof: Let (S, T_{deg}) be a degree-topological space and let T_{deg} be a T_0 space. Then, any distinct elements v_1 and v_2 in S there exists an open set W in T_{deg} such that $v_1 \in W, v_2 \notin W$ or $v_2 \in W, v_1 \notin W$. Therefore, T_{deg} is a PT_0-space.■

Remark 4.10 The converse of Theorem 4.9 is not true in general for a degree-topology that is generated by a complete graph which is a PT_0-space due to Theorem 4.8, but it is not a T_0 space.

Remark 4.11

1. The degree-topology of a connected graph is not necessarily satisfied a PT_0-separation axiom as an example, the degree-topology generated by the path graph P_n with order $n \neq 3$ which is a connected graph but it does not satisfy the PT_0-separation axiom as shown in Example 4.3.
2. The degree-topology of a disconnected graph is not necessarily satisfied the PT_0-separation axiom as shown by the following example:

Example 4.12 Let $G(V,E)$ be a disconnected graph which is shown by a path graph of order four with an isolated vertex v_5. Then $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $T_{deg}(G) = \{\emptyset, V, \{v_1, v_4\}, \{v_2, v_3\}, \{v_5\}\}$. We have, $T_{deg}(G)$ does not satisfy the PT_0-separation axiom since $v_2, v_3 \in V(G)$, but there is no open set in $T_{deg}(G)$ that includes just one of them. Indeed, there is no v_2, v_3- path of length greatest or equal to two.

Remark 4.13 The degree-topological space is not necessarily to be a PT_0- space, since the degree-topology generated by the path graph P_5 with the order $n = 5$, did not satisfy the PT_0-separation axiom, as shown in Example 4.3.

Theorem 4.14 The degree-topology generated by a complete bipartite graph $K_{n,m}$ with $n \neq m$ vertices satisfies the PT_0-separation axiom.

Proof: Assume that $K_{n,m}(V,E)$ is a complete bipartite graph with $n \neq m$ and the vertex set V. Let V be partitioned into two disjoint sets V_1 and V_2, such that the number of vertices for V_1 and V_2 is n and m, respectively. By definition of the complete bipartite graph, every vertex in V_1 is adjacent to all vertices in V_2. Consequently, there are three cases:

- If the distinct vertices v_i and v_j in V_1. Then, it has a $v_i v_j$- path of the length equals two, that is v_i, v_k, v_j for any vertex v_k in V_2, where $i, j = 1,2, \ldots, n$ and $k = 1,2, \ldots, m$.
- If the distinct vertices v_r and v_s in V_2. Then, it has a $v_r v_s$- path of the length equals two, that is v_r, v_z, v_s for any vertex v_z in V_1, where $r, s = 1,2, \ldots, m$ and $z = 1,2, \ldots, n$.
- If the vertices v_i in V_1 and v_j in V_2. So, there exists an open set in $T_{deg}(K_{n,m})$ contains one of them but not the other.

Thus, the degree-topology generated by a complete bipartite graph $K_{n,m}$ with $n \neq m$ satisfies a PT_0-separation axiom, see Figure 11.■

Theorem 4.15 The degree-topology generated by a complete bipartite graph $K_{n,m}$ with $n = m \geq 2$, vertices satisfies a PT_0-separation axiom.

Proof: Assume that $K_{n,m}(V,E)$ is a complete bipartite graph with $n = m \geq 2$ and the vertex set V. Let V be partitioned into two disjoint sets V_1 and V_2 and the number of vertices for V_1 and V_2 is n. By definition of the complete bipartite graph, every vertex in V_1 is adjacent to all vertices in V_2, we have the following three cases:

- If the vertices v_i in V_1 and v_j in V_2. Thus, there exist vertices v_r in V_1 and v_s in V_2 such that a $v_i v_j$- path of the length equal to three is that v_i, v_r, v_s, v_j, where $r, i, j, s = 1,2, \ldots, n$ and $r \neq i \neq j \neq s$.
- If the distinct vertices v_i and v_j in V_1. So, it has a $v_i v_j$- path of the length equals two, that is v_i, v_k, v_j for any vertex v_k in V_2, where $i, j = 1,2, \ldots, n$ and $k = 1,2, \ldots, m$.
- If the distinct vertices v_r and v_s in V_2. Then, it has a $v_r v_s$- path of the length equals two, that is v_r, v_z, v_s for any vertex v_z in V_1, where $r, s = 1,2, \ldots, m$ and $z = 1,2, \ldots, n$.

Thus, the degree-topology generated by complete bipartite graph $K_{n,m}$ with $n = m \geq 2$ satisfies a PT_0-separation axiom.■

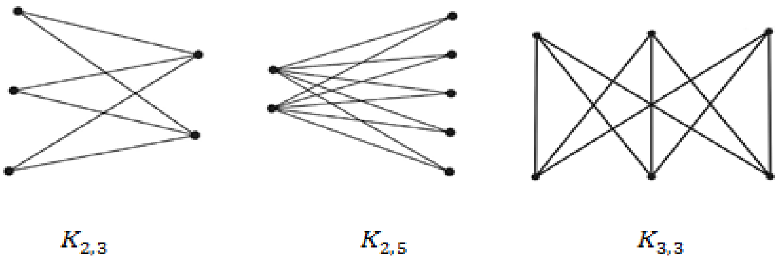

Figure 11: Examples of The complete bipartite graph $K_{2,3}, K_{2,5}$ and $K_{3,3}$

Theorem 4.16 The degree-topology generated by wheel graph W_n with $n \geq 4$ satisfies a PT_0-separation axiom**.**

Proof: Let $W_n(V, E)$ be the wheel graph of order n with $n \geq 4$, such that $V(W_n) = \{v_1, v_2, v_3, \dots, v_{n-1}, v_n\}$ where v_n is the universal vertex and $\{v_1, v_2, \dots, v_{n-1}\}$be a vertex of the cycle graph C_{n-1} in W_n. If $n = 4$, and $T_{\deg}(W_4)= \{V, \emptyset\}$, we have the two cases:

- If the distinct vertices v_4 and v_i with $i = 1,2,3$. Then, it has a v_4v_i- path of the length equals two, that is v_4, v_j, v_i for any vertex v_j with $j = 1,2,3$ and $i \neq j$.
- If the distinct vertices v_i and v_j in $\{v_k\}_{k=1}^{3}$. Then, it has a v_iv_j- path of the length equals two, that is v_i, v_4, v_j.

Now, if $n > 4$ and $T_{\deg}(W_n)= \{V, \emptyset, \{v_n\}, \{v_i\}_{i=1}^{n-1}\}$ then, for any distinct vertices v_n and v_i, with $i = 1,2, \dots, n-1$, there is an open set $\{v_n\}$, in $T_{deg}(W_n)$ that includes just one of them. By definition of wheel graph, v_n that join with all vertices of $C_{n-1}$by an edge then, for vertices v_i and v_j in $V(C_{n-1})$ with $i, j = 1,2, \dots n-1$, $i \neq j$. there is $P_{v_iv_j}$ with $|P_{v_1v_3}| = 2$, that is $v_iv_nv_j$. Hence, $T_{\deg}\,(W_n)$ with $n \geq 4$ satisfies the PT_0-separation axiom.■

Proposition 4.17 The degree-topology generated by helm graph H_5 satisfies the PT_0-separation axiom**.**

Proof: Let $H_5(V, E)$ be the helm graph of order 9, where $V(H_5)$=$\{v_1, v_2, v_3, v_4, v_5, u_2, u_3, u_4\}$ where v_5 is the vertex that joins with cycle vertices $\{v_1, v_2, v_3, v_4\}$ in H_5 and $\{u_1, u_2, u_3, u_4\}$ be pendent vertices that join with cycle vertices. Assume that u_1, u_2, u_3 and u_4 join with $v_1, v_2, v_3$and v_4, respectively. Since $T_{deg}(H_5)$=$\{V, \emptyset, \{v_1, v_2, v_3, v_4, v_5\}, \{u_1, u_2, u_3, u_4\}\}$, then we have the following cases:

- If the distinct vertices u_i and v_j with $j = 1,2,3,4,5$ and $i = 1,2,3,4$. So, there exists an open set in $T_{deg}(H_5)$ contains one of them but not the other.
- If the distinct vertices v_i and v_j in $\{v_k\}_{k=1}^{4}$. Then, it has a v_iv_j- path of the length equals two, that is v_i, v_5, v_j.
- If the distinct vertices v_5 and v_i with $i = 1,2,3,4$. Then, it has a v_5v_i- path of the length equals two, that is v_5, v_j, v_i for some vertex v_j with $j = 1,2,3,4$ and $i \neq j$.
- If the distinct vertices u_i and u_j in $\{u_k\}_{k=1}^{4}$. Then, there exists v_i and v_j in $\{v_k\}_{k=1}^{4}$ with $i \neq j$ such that v_i and v_j are adjacent with u_i and u_j, respectively. Then it has a u_iu_j- path of the length greatest than two, that is u_i, v_i, v_5, v_j, u_j.

Thus, the degree-topology generated by a helm graph H_5 satisfies the PT_0-separation axiom.■

Theorem 4.18 The degree-topology generated by helm graph H_n with $n = 4$ or $n > 5$ satisfies the PT_0-separation axiom**.**

Proof: Assume that $H_n(V, E)$ is a helm graph of order $2n - 1$ and $W_n(V, E)$ be a wheel graph in helm graph with $n = 4$ or $n > 5$. By definition, the wheel graph contains a cycle graph C_{n-1} and the vertex v_n connected with all vertices of C_{n-1}such that $V(C_{n-1})$=$\{v_1, v_2, \dots v_{n-1}\}$. By definition of the helm graph, there is $\{u_1, u_2, \dots, u_{n-1}\}$ pendent vertices that join with cycle vertices. So, $V(H_n)$=$\{v_i\}_{i=1}^{n-1} \cup \{u_i\}_{i=1}^{n-1} \cup \{v_n\}$. Since $T_{\deg}(H_n)$=$\{V, \emptyset, \{v_n\}, \{v_i\}_{i=1}^{n-1}, \{u_i\}_{i=1}^{n-1}, \{v_i\}_{i=1}^{n}, \{v_n, u_i\}_{i=1}^{n-1}\}, \{u_i, v_i\}_{i=1}^{n-1}\}$, then we have the following cases:

- If the distinct vertices v_n and v_i with $i = 1,2,\dots,n-1$. Then there exists an open set in $T_{deg}(H_n)$ contains one of them but not the other.
- If the distinct vertices v_n and u_j with $j = 1,2,\dots,n-1$. So, there exists an open set in $T_{deg}(H_n)$ contains just one.
- If the distinct vertices u_i and v_j with $i,j = 1,2,\dots,n-1$. Then, there exists an open set in $T_{deg}(H_n)$ contains one of them but not the other.
- If v_i and v_j in $\{v_k\}_{k=1}^{n-1}$ with $i \neq j$. Then, it has a $v_i v_j$- path of the length equals two, that is v_i, v_n, v_j.
- If the distinct vertices u_i and u_j in $\{u_k\}_{k=1}^{n-1}$. Then, there exist vertices v_r and v_s in $\{v_k\}_{k=1}^{n-1}$ with $r \neq s$ such that v_r and v_s are adjacent with u_i and u_j, respectively. Then there is $P_{u_i u_j}$ with $\left|P_{u_i u_j}\right| \geq 2$, that is u_i, v_r, v_n, v_s, u_j.

Thus, the degree-topology generated by a helm graph H_n with $n = 4$ or $n > 5$ satisfies the PT_0-separation axiom.■

Proposition 4.19 The degree-topology generated by gear graph G_4 satisfies the PT_0-separation axiom.

Proof: Let $G_4(V,E)$ be the gear graph of order 7, where $V(G_4) = \{v_1, v_2, v_3, v_4, u_1, u_2, u_3\}$ where v_4 is the vertex that joins with cycle vertices $\{v_1, v_2, v_3\}$ in $G_4$and $\{u_1, u_2, u_3\}$ be a vertex that is added between each pair of adjacent vertices in the outer of the cycle.Since $T_{\text{deg}}(G_4)$=$\{V, \emptyset, \{v_1, v_2, v_3, v_4\}, \{u_1, u_2, u_3\}\}$, then we have the following cases:

- If the distinct vertices u_i and v_j with $i = 1,2,3,4$ and $j = 1,2,3$. So, there exists an open set in $T_{deg}(G_4)$ contains one of them but not the other.
- If the distinct vertices v_i and v_j in $\{v_k\}_{k=1}^{3}$. Then, it has a $v_i v_j$- path of the length equals two, that is v_i, v_4, v_j.
- If the distinct vertices v_4 and $v_r$with $r = 1,2,3$. Let the vertex u_i adjacent with v_r and another vertex in $\{v_k\}_{k=1}^{3}$ say v_s with $s \neq r$ and $i = 1,2,3$. Then, there is $P_{v_4 v_r}$ with $\left|P_{v_4 v_r}\right| \geq 2$, that is v_r, u_i, v_s, v_4.
- If the distinct vertices u_i and $u_j$with $i,j = 1,2,3$. Let a vertex v in $\{v_k\}_{k=1}^{3}$ adjacent with u_i and u_j. Then it has a $u_i u_j$- path of the length equals two, that is u_i,v, u_j.

Thus, the degree-topology generated by a gear graph G_4 is satisfying the PT_0-separation axiom.■

Theorem 4.20 The degree-topology generated by gear graph G_n with $n > 4$ satisfies the PT_0-separation axiom.

Proof: Let $G_n(V,E)$be the gear graph of order $2n-1$, and $W_n(V,E)$ be a wheel graph in gear graph with $n > 4$. By definition, the wheel graph contains a cycle graph C_{n-1} and the vertex v_n connected with all vertices of C_{n-1}such that $V(C_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$. By definition of the gear graph, there is $\{s_1, s_2, .., s_{n-1}\}$ vertices that are added between each pair of adjacent vertices in the outer of the cycle. So, $V(G_n) = \{v_i\}_{i=1}^{n-1} \cup \{s_i\}_{i=1}^{n-1} \cup \{v_n\}$. Since $T_{deg}(G_n) = \{V, \emptyset, \{v_n\}, \{v_i\}_{i=1}^{n-1}, \{s_i\}_{i=1}^{n-1}, \{v_i\}_{i=1}^{n}, \{v_n, s_i\}_{i=1}^{n-1}, \{v_i, s_i\}_{i=1}^{n-1}\}$, then we have the following cases:

- If the distinct vertices s_i and v_j with $i,j = 1,2,\dots,n-1$. So, there exists an open set in $T_{deg}(G_n)$ contains just one of them.

- If the distinct vertices v_n and s in $\{s_k\}_{k=1}^{n-1}$. Then, there exists an open set in $T_{deg}(G_n)$ contains one of them but not the other.
- If the distinct vertices v_n and v in $\{v_k\}_{k=1}^{n-1}$. Then, there exists an open set in $T_{deg}(G_n)$ contains only one of them.
- If the distinct vertices v_i and v_j in $\{v_k\}_{k=1}^{n-1}$. Then, it has a $v_i v_j$- path of the length equals two, that is v_i, v_n, v_j.
- If the distinct vertices s_i and s_j with $i,j = 1,2,\ldots,n-1$. If s_i and s_j be adjacent to a vertex v in $\{v_k\}_{k=1}^{n-1}$.Then, there is $P_{s_i s_j}$ with $\left|P_{s_i s_j}\right| = 2$, that is s_i, v, s_j. If s_i and s_j are non-adjacent to a vertex v in $\{v_k\}_{k=1}^{n-1}$. Then, there exist distinct vertices v_r and v_s in $\{v_k\}_{k=1}^{n-1}$ such that v_r join with s_i and v_s join with s_j by an edge. Then, there is $P_{s_i s_j}$ with $\left|P_{s_i s_j}\right| \geq 2$, that is s_i, v_r, v_n, v_s, s_j.

Thus, the degree-topology generated by a gear graph G_n with $n > 4$ satisfies the PT_0-separation axiom.■

Theorem 4.21 The degree-topology generated by pan graph N_n with $n \geq 3$ is satisfying the PT_0-separation axiom.

Proof: Let $N_n(V,E)$ be a pan graph of order $n+1$ and $V(N_n)$ ={$v_1, v_2, .., v_n, v_{n+1}$}. Assume $C_n(V,E)$ be a cycle graph in a pan graph N_n with $n \geq 3$ where $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. By definition of the pan graph, there is a vertex v_{n+1} connected with one vertex of C_n say v_k. Since $T_{deg}(N_n)$=$\{V, \emptyset, \{v_k\}, \{v_{n+1}\}, \{v_i\}_{i=1,i\neq k}^{n}, \{v_k, v_{n+1}\}, \{v_i\}_{i=1}^{n}, \{v_i\}_{i=1,i\neq k}^{n+1}\}$ with $i \neq k$,we have the following cases:

- For the distinct vertices v_i in $\{v_i\}_{i=1}^{n}$ and v_{n+1}, there exists an open set $\{v_{n+1}\}$ in $T_{deg}(N_n)$ such that $v_{n+1} \in \{v_{n+1}\}$ and $v_i \notin \{v_{n+1}\}$.
- For the distinct vertices v_k and v_i in $\{v_k\}_{k=1}^{n}$ with $i \neq k$ there exists an open set $\{v_i\}_{i=1}^{n}$ in $T_{deg}(N_n)$ such that $v_i \in \{v_i\}_{i=1}^{n}$and $v_k \notin \{v_i\}_{i=1}^{n}$.

- If the distinct vertices v_i and v_j with $i,j = 1,2,\ldots,n$ such that $i \neq j \neq k$. Then there exists one or more than one vertex v_r in $\{v_k\}_{k=1}^{n}$, $r \neq i,j$. If there is one vertex,v_r then v_r adjacent to a vertex v_i and v_j. Then, there is $P_{v_i v_j}$ with $\left|P_{v_i v_j}\right| = 2$, that is v_i, v_r, v_j. Otherwise, there is more than one sequential vertex in $\{v_k\}_{k=1}^{n}$ such that one of them adjacent to v_i or v_j.

Hence, the degree-topology generated by a pan graph N_n with $n \geq 3$ satisfies the PT_0-separation axiom.■

Theorem 4.22 The degree-topology generated by windmill graph W_n^m where n is the order of a complete graph K_n in W_n^m and m is the number of copies for K_n satisfies the PT_0-separation axiom.

Proof: Let $W_n^m(V,E)$ be a windmill graph of order $m(n-1)+1$ where n is the order of a complete graph $K_n(V,E)$ and m is the number of copies K_n. Assume that $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and v be a vertex that common in all copies of K_n. Since $T_{\deg}(W_n^m)$= {V, $\emptyset$, $\{v\}$, $\bigcup_{i=1}^{m}\{v_1^i, \ldots v_{n-1}^i\}$}, then there are two cases:

- For the distinct vertices v_k in $\bigcup_{i=1}^{m}\{v_1^i, \ldots v_{n-1}^i\}$ and v, there exists an open set $\{v\}$ in $T_{deg}(W_n^m)$ such that $v \in \{v\}$ and $v_k \notin \{v\}$.

- For the distinct vertices v_r and v_s in $\bigcup_{i=1}^{m}\{ v_1^i,...v_{n-1}^i\}$ with $r \neq s$. Then, there is $P_{v_r v_s}$ with $|P_{v_r v_s}| = 2$, that is v_r, v_n, v_s.

Hence, the degree-topology generated by a windmill graph W_n^m satisfies the PT_0-separation axiom. ■

5. Conclusions

To sum up, we are to investigate the degree-topology of the wheel, helm, gear, pan, tadpole, and windmill graphs, respectively. Additionally, a new separation axiom is known as PT_0-space was introduced that connects paths in the graph with topological space. Further, PT_0-space studied for the above topologies.

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