DOI: 10.24996/ijs.2024.65.9.31

ISSN: 0067-2904

First Integrals and a Zero-Hopf Bifurcation of the Four-Dimensional Lotka-Volterra Systems

Sirwan A. Mustafa*, Niazy H. Hussen
Department of Mathematics, Faculty of Science, Soran University, Erbil, Iraq

Received: 4/5/2023 Accepted: 11/8/2023 Published: 30/9/2024

Abstract
In this paper, the integrability and a zero-Hopf bifurcation of the four-dimensional Lotka-Volterra systems are studied. The requirements for this kind of system's integrability and a line of singularities with two zero eigenvalues are provided. We identify the parameters that lead to a zero-Hopf equilibrium point at each point along the line of singularities. We show that there is only one parameter that displays such equilibria. The first-order averaging method is also employed, although this method will not give any information about the bifurcate periodic solutions that bifurcate from the zero-Hopf equilibria.

Keywords: Lotka-Volterra system, Invariant algebraic hypersurfaces, Darboux first integral, Zero-Hopf bifurcation, Averaging theory.

التكاملات الأولى والتشعب الصفري لأنظمة لوتكا فولتيرا رباعية الأبعاد

سيروان انور مصطفى *, نيازى هادى حسين
قسم الرياضيات، كلية العلوم، جامعة سوران، أربيل، العراق

الخلاصة
في هذا البحث تم دراسة قابلية التكامل والتشعب الصفري هوبف لأنظمة لوتكا–فولتيرا رباعية الأبعاد. يتم إعطاء شروط تكامل هذا النوع من النظام وخط التفردات مع قيمتين من قيم الذاتية. نحدد المعلمات التي تؤدي إلى نقطة توازن صفرية لهوبف في أي نقطة على طول خط التفردات. نظهر أن هناك متغير واحد فقط يعرض مثل هذا التوازن. يتم أيضًا استخدام طريقة حساب المتوسط من الدرجة الأولى ، على الرغم من أن هذه الطريقة لا توفر أي معلومات حول الحلول الدورية ذات التشعبين التي تتشعب من توازن صفر هوبف.

1. Introduction

Lotka -Volterra systems describing the interaction of n species were introduced independently by Alfred Lotka (1925) and Vito Volterra (1926) in the theory of biological populations. They consist of n first-order differential equations

$$\frac{dx_i(t)}{dt} = x_i(t)\big(b_i + \sum_{j=1}^{n} a_{ij}\, x_j(t)\big), \qquad i = 1, \dots, n, \tag{1}$$

where $x_i(t)$ is the number of individuals in the $i\ th$ population at time t, b_i is the growth rate of the $i\ th$ population and a_{ij} are the interaction coefficients of the species [1].

*Email: hassanalrdeny308@gmail.com

The system described by equation (1) is called a competitive system. This system is a basic model of predator-prey interactions. The Lotka–Volterra system in $\mathbb{R}^4$ with coordinates (x, y, z, w) is a quadratic polynomial differential system of the form

$$\begin{cases} \dot{x} = x(b_1 + a_{11}x + a_{12}y + a_{13}z + a_{14}w), \\ \dot{y} = y(b_2 + a_{21}x + a_{22}y + a_{23}z + a_{24}w), \\ \dot{z} = z(b_3 + a_{31}x + a_{32}y + a_{33}z + a_{34}w), \\ \dot{w} = w(b_4 + a_{41}x + a_{42}y + a_{43}z + a_{44}w), \end{cases} \tag{2}$$

where the dot denotes the derivative with respect to the independent variable t which is usually called the time and $b_i > 0$ and $a_{ij} < 0$ $(i, j = 1, \dots, 4)$ are real parameters, see [2].

The general Lotka-Volterra model has become the starting point for a wide variety of mathematical models in ecology, physics, economics, etc. [3].

Various investigations of system (1) have been studied by numerous authors, including, Wang and Xiao [4] studied Hopf-bifurcation for the four-dimensional Lotka-Volterra system by using simulations and the linearization technique, Lyu and Jablonski [5] investigated the four-dimensional discrete-time Lotka-Volterra model with the use of a practical ecological system, Kowgier [6] showed on a few models how the survival probability of four populations alters with the assumption that they reach an equilibrium level determined by the same number of individuals, Farhan et al. [7] investigated the stability of the four-dimensional Lotka-Volterra model, and Antonov et al. [8] determined some criteria for the existence of the first integrals of the prey-predator tridiagonal 4-dimensional Lotka-Volterra models.

The purpose of this paper is to study two objectives: The first main objective of this paper is to advance our understanding of the complexity of system (2), more specifically, the dynamics of the system and this is done by examining its integrability. Furthermore, within the class of first integrals, the simpler ones are known as Darboux first integrals in $\mathbb{R}^4$. For more details, see [9, 10, 11, 12].

The second objective of this research is to investigate a zero-Hopf bifurcation at the zero-Hopf equilibrium point. We recall that a zero-Hopf equilibrium point is an equilibrium that has a pair of purely imaginary and two zero eigenvalues. When an infinitesimal periodic orbit bifurcates from the equilibrium point, such a kind of bifurcation is called zero-Hopf bifurcation. This type of bifurcation has been studied by [13, 14, 15, 16, 17]. It has been shown that, the isolated zero-Hopf equilibrium point of some complicated invariant sets may be bifurcated under suitable conditions. In [18, 19, 20, 21] the authors obtain some investigations as a chaotic behavior. The averaging method is a classical and useful computational technique for analysing nonlinear oscillations. It has been used by many authors to study the bifurcating periodic orbits from a zero-Hopf equilibrium point. The first order of the averaging method is used in the work [22, 23, 24, 25]. There are some works on a zero-Hopf bifurcation of the four-dimensional systems, Jaume Llibre and Yuzhou Tian in [26], Jaume Llibre et al. in [27]. Furthermore, the authors of [28, 29, 30, 31, 32] investigated periodic orbits in $\mathbb{R}^4$ using the averaging method.

This paper is organized as follows. In section 2, the Darboux theory of integrability and the line of singularities of the 4DLVS are studied. In section 3, the precise parametric requirements for a 4DLVS zero-Hopf equilibrium are given. We explain the averaging method of the first order. Finally, the conclusion of this work is given in Section 4.

2. Darboux theory

In this section, the integrability and the existence of a line of singularities are studied for 4DLVS. Some conditions are established in order to construct the invariant algebraic surfaces. In addition, the sufficient conditions for the existence of a line of singularity with two zero eigenvalues are obtained. In these conditions, a function of the Darboux type produces three linearly independent first integrals of the 4DLVS. We denote by $\mathbb{C}[x,y,x,w]$ the ring of polynomials in the variables x,y,z and w and coefficients are in $\mathbb{C}$. Given $f \in \mathbb{C}[x,y,z,w]$ is a Darboux polynomial of system (2) if there exists $K \in \mathbb{C}[x,y,z,w]$ called the cofactor such that

$$\dot{x}\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial y} + \dot{z}\frac{\partial f}{\partial z} + \dot{w}\frac{\partial f}{\partial w} = K\,f \quad (3)$$

The degree of K of system (2) is at most one. When a real polynomial system contains a complex Darboux polynomial, it also contains its conjugate. It is critical to study the complex Darboux polynomials of real polynomial differential systems because sometimes they force the real integrability of the system, [33]. It is worth noting that we may write f as:

$$\chi_f = Kf, \quad \text{where} \quad \chi = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z} + \dot{w}\frac{\partial}{\partial w}. \quad (4)$$

A non-locally constant C^1 function $H: U \to \mathbb{R}$ is called a first integral of system (2) on $U \subset \mathbb{R}^4$ if $H(x(t), y(t), z(t), w(t))$ is constant for all of the values of t for which $(x(t), y(t), z(t), w(t))$ is a solution of system (2) contained in U, [33]. If system (2) has three distinct first integrals, it is integrable.

To find a first integral in Darboux type, we shall find enough invariant algebraic surfaces. So, we state the following propositions of system (2).

Proposition (2.1): System (2) always has four Darboux polynomials, $f_1 = x, f_2 = y, f_3 = z$ and $f_4 = w$ with cofactors $k_1 = b_1 + a_{11}x + a_{12}y + a_{13}z + a_{14}w, k_2 = b_2 + a_{21}x + a_{22}y + a_{23}z + a_{24}w$, $k_3 = b_3 + a_{31}x + a_{32}y + a_{33}z + a_{34}w$ and $k_4 = b_4 + a_{41}x + a_{42}y + a_{43}z + a_{44}w$, respectively.

Proof: Clearly, $\chi(f_i) = k_i f_i$ where f_i and $k_i, i = 1, \cdots, 4$ are defined in the proposition. Therefore, $f_i = 0, i = 1, \cdots, 4$ are Darboux polynomials of system (2).

Proposition (2.2): The function $f_5(x,y,z,w) = 1 - (x+y+z+w)$ is Darboux polynomial of system (2) with cofactor $k_5 = -(b_1x + b_2y + b_3z + b_4w)$ if and only if the following conditions are satisfied:

$$a_{ij} = -(b_i + b_j + a_{ji}), \text{ and } a_{ii} = -b_i, \text{ for } j > i \text{ and } i,j = 1,2,3,4. \quad (5)$$

Proof: Firstly, we consider that the function $f_5(x,y,z,w) = 1 - (x+y+z+w)$ is Darboux polynomial of system (2), then from the equation (4), the set of conditions (5) is obtained. Conversely, if the conditions (5) satisfied, directly $\chi(f_5) = k_5 f_5$. Thus, $f_5(x,y,z,w) = 0$ is Darboux polynomial of system (2).

Proposition (2.3): The function $f_6(x,y,z,w) = (1-x-y)(x+y+z+w-1)$ is Darboux polynomial of system (2) with cofactor $k_6 = 2a_{41}x + (-2a_{31} + 2a_{32} + 2a_{41})y + (a_{31} - a_{41})z$ if and only if the conditions (5) with the following conditions hold:

$$a_{42} = a_{32} - a_{31} + a_{41},\ b_1 = -a_{41},\ b_2 = a_{31} - a_{32} - a_{41},\ b_3 = a_{41} - a_{31} \text{ and } b_4 = 0. \quad (6)$$

Proof: Firstly, we assume that the function $f_6(x,y,z,w) = 0$ is Darboux polynomial of system (2), then from equation (4), the set of conditions (5) and (6) are obtained. Conversely,

if the conditions (5) and (6) hold, then it is easy to show that $\chi(f_6) = k_6 f_6$. Thus, the function $f_6(x, y, z, w) = 0$ is Darboux polynomial of system (2).
The following theorem is the first main result of this work.

Theorem (2.4): For the four-dimensional Lotka-Volterra system (2) satisfying conditions (5) and (6) the following results are obtained
1. System (2) has a line of singularities with two zero eigenvalues.
2. System (2) is integrable. Accurately, the system has three independent first integrals.

Proof: We can rewrite system (2) with conditions (5) and (6) of the following

$$\begin{cases} \dot{x} = x(a_{41}x - (a_{21} + a_{31} - a_{32} - 2a_{41})y - a_{41}), \\ \dot{y} = y(a_{21}x - (a_{31} - a_{32} - a_{41})y + a_{31} - a_{32} - a_{41}), \\ \dot{z} = z(a_{31}x + a_{32}y + (a_{31} - a_{41})z + (a_{31} - a_{41} - a_{43})w - a_{31} + a_{41}), \\ \dot{w} = w(a_{41}x - (a_{31} - a_{32} - a_{41})y + a_{43}z). \end{cases} \tag{7}$$

The singular points of system (7) can be found by solving the following equations:

$$\begin{cases} 0 = x(a_{41}x - (a_{21} + a_{31} - a_{32} - 2a_{41})y - a_{41}), \\ 0 = y(a_{21}x - (a_{31} - a_{32} - a_{41})y + a_{31} - a_{32} - a_{41}), \\ 0 = z(a_{31}x + a_{32}y + (a_{31} - a_{41})z + (a_{31} - a_{41} - a_{43})w - a_{31} + a_{41}), \\ 0 = w(a_{41}x - (a_{31} - a_{32} - a_{41})y + a_{43}z). \end{cases}$$

A simple analysis, using Maple program, directly obtains the following solutions to the above system of equations which are singular points of system (7)
$(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,w)$,
$\left(1,0,-\frac{a_{41}}{a_{43}},\frac{a_{41}}{a_{43}}\right)$, $\left(1,0,\frac{a_{41}}{a_{41}-a_{31}},0\right)$, $(0,1,\frac{a_{32}+a_{41}-a_{31}}{a_{41}-a_{31}},0)$,
$\left(0,1,-\frac{a_{32}+a_{41}-a_{31}}{a_{43}},\frac{a_{32}+a_{41}-a_{31}}{a_{43}}\right)$, $(-\frac{a_{32}+a_{41}-a_{31}}{a_{41}-a_{21}},\frac{a_{41}}{a_{41}-a_{21}},\frac{a_{32}+a_{41}-a_{31}-a_{21}}{a_{41}-a_{21}},0)$,
where $a_{43} \neq 0$ and $a_{41} - a_{31} \neq 0$, system (2) has the following line of singularities,

$$L = \left\{(x, y, z, w) \in \mathbb{R}^4 : (x, y, z, w) = (\frac{a_{31}-a_{32}-a_{41}}{a_{41}-a_{21}}, \frac{a_{41}}{a_{41}-a_{21}}, 0, t)\right\}, \tag{8}$$

where $t \in \mathbb{R}$, such that $a_{41} - a_{21} \neq 0$.
In order to prove the second part of the theorem, we try to construct Darboux first integral of the form

$$H = \Pi_{i=1}^{6} f_i^{\lambda_i},$$

where f_i are Darboux polynomials of system (2) and their cofactor k_i are defined in Propositions (2.1), (2.2) and (2.3). Using the form $\sum_{i=1}^{6} \lambda_i k_i = 0$, where $\lambda_i \in \mathbb{C}$, we have the following equation

$$\begin{aligned} &(a_{21}\lambda_2 + a_{31}\lambda_3 + a_{41}\lambda_1 + a_{41}\lambda_4 + a_{41}\lambda_5 + 2a_{41}\lambda_6)x + ((a_{32} - a_{21} - a_{31} + 2a_{41})\lambda_1 \\ &+(a_{32} - a_{31} + a_{41})\lambda_2 + a_{32}\lambda_3 + (a_{32} - a_{31} + a_{41})\lambda_4 + (a_{32} - a_{31} + a_{41})\lambda_5 + (2a_{32} \\ &-2a_{31} + 2a_{41})\lambda_6)y + \left((a_{31} - a_{41})\lambda_3 + a_{43}\lambda_4 - (a_{41} - a_{31})\lambda_5 + (a_{31} - a_{41})\lambda_6\right)z \\ &+(a_{31} - a_{41} - a_{43})\lambda_3 w - a_{41}\lambda_1 + (a_{31} - a_{32} - a_{41})\lambda_2 + (a_{41} - a_{31})\lambda_3 = 0. \end{aligned}$$

This gives that

$$
\begin{aligned}
0 &= (a_{31} - a_{41} - a_{43})\lambda_3,\\
0 &= -a_{41}\lambda_1 + (a_{31} - a_{32} - a_{41})\lambda_2 + (a_{41} - a_{31})\lambda_3,\\
0 &= a_{21}\lambda_2 + a_{31}\lambda_3 + a_{41}\lambda_1 + a_{41}\lambda_4 + a_{41}\lambda_5 + 2\,a_{41}\lambda_6,\\
0 &= (a_{31} - a_{41})\lambda_3 + a_{43}\lambda_4 - (a_{41} - a_{31})\lambda_5 + (a_{31} - a_{41})\lambda_6,\\
0 &= (a_{32} - a_{21} + 2\,a_{41} - a_{31})\lambda_1 + (a_{32} - a_{31} + a_{41})\lambda_2 + a_{32}\lambda_3\\
&\quad +(a_{32} - a_{31} + a_{41})\lambda_4 + (a_{32} - a_{31} + a_{41})\lambda_5 + (2\,a_{32} - 2\,a_{31} + 2\,a_{41})\lambda_6.
\end{aligned} \tag{9}
$$

For the above equations, there is a set of solutions indicates that there exist $\lambda_i \in \mathbb{R}$ such that $\sum_{i=1}^{n} \lambda_i k_i = 0$, λ_i not all zero. Thus, the system has a first integral of Darboux type. We choose the following solutions for equation (9),

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (\eta_1, 1,0, (a_{41} - a_{31})\eta_2, a_{43}\eta_2, 0),$$
$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (0,0,0, \eta_3, \eta_4, 1),$$
$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (\eta_1, 1,0, (a_{41} - a_{31})\eta_2, \eta_5, 1),$$

where $\eta_1 = \frac{a_{31}-a_{32}-a_{41}}{a_{41}}$, $\eta_2 = \frac{(a_{21}+a_{31}-a_{32}-a_{41})}{a_{41}(a_{31}-a_{41}-a_{43})}$, $\eta_3 = \frac{-a_{31}+a_{41}}{a_{31}-a_{41}-a_{43}}$, $\eta_4 = \frac{a_{41}-a_{31}+2a_{43}}{a_{31}-a_{41}-a_{43}}$ $\eta_5 = \frac{(a_{21}+a_{31}-a_{32}+a_{41})a_{43}-a_{41}(a_{31}-a_{41})}{a_{41}(a_{31}-a_{41}-a_{43})}$ with $a_{41} \neq 0$ and $a_{31} - a_{41} - a_{43} \neq 0$. Consequently, the following functions are first integrals of the system,

$$H_1 = x^{\eta_1} y w^{(a_{41}-a_{31})\eta_2} (1 - x - y - z - w)^{a_{43}\eta_2},$$
$$H_2 = w^{\eta_3} (x + y - 1)(1 - x - y - z - w)^{\frac{a_{43}}{a_{31}-a_{41}-a_{43}}},$$
$$H_3 = x^{\eta_1} y w^{(a_{41}-a_{31})\eta_2} (x + y - 1)(1 - x - y - z - w)^{a_{43}\eta_4}.$$

It is simple to verify that ∇H_1, ∇H_2 and ∇H_3 are linearly independent, hence the first three integrals H_i, $i = 1,2,3$ are independent. This means that system (2) with conditions (5) and (6) is integrable.

3. Zero-Hopf Bifurcation

We here investigate a zero-Hopf bifurcation of the 4DLVS via the first-order averaging method. Also, we prove that system (2) has only one line of singularity as a zero-Hopf equilibrium. This section is organized as follows: In the first subsection, we give the first-order averaging method and some related concepts. The second subsection, we use the first-order averaging theory to illustrate that there are no periodic solutions that bifurcate from the zero-Hopf equilibrium point located at that line of singularity of system (2).

3.1 First-Order Averaging Method

The averaging method is one of the most significant theories for predicting periodic solutions for various differential systems. Several authors have devoted their efforts to analyzing the existence of periodic solutions via the averaging method as we see in the work of Sanders and Murdock [34], McCracken and Marsden [35], Chow [36], Buica et al. [37]. In the following, we consider the perturbation differential systems

$$\dot{x} = F_0(t, x) + \epsilon F_1(t, x) + \epsilon^2 F_2(t, x, \epsilon), \tag{10}$$

where ϵ is too small positive perturbation parameter, $F_0: \mathbb{R} \times U \longrightarrow \mathbb{R}^n, F_1: \mathbb{R} \times U \longrightarrow \mathbb{R}^n$, and $F_2 : \mathbb{R} \times U \times (-\epsilon_0, \epsilon_0) \longrightarrow \mathbb{R}^n$ are C^2 functions which are T-periodic in t, and $U \subset \mathbb{R}^n$. The unperturbed part of system (10) is

$$\dot{x} = F_0(t, x). \tag{11}$$

The existence of a submanifold of periodic solutions for system (11) is supposed. This means that all solutions are T-periodic. We write the linearization of the unperturbed system (11) along a periodic solution $x(t, \varphi)$ satisfies the initial condition $x(0, \varphi) = \varphi$ as

$$\dot{y} = D_x F_0(t, x(t, \varphi))y, \tag{12}$$

where the Jacobian matrix of F_0 with respect to x is $D_x F_0$, and here we denote the fundamental matrix of system (12) by $M_\varphi(t)$. Also, we suppose that there is an open set W with $Cl(W) \subset U$ such that for each $\varphi \in Cl(W), x(t,\varphi)$ is a T-periodic.

Theorem (3.1.1): We assume that the function $\mathcal{F}: Cl(W) \longrightarrow \mathbb{R}^n$

$$\mathcal{F}(\varphi) = \frac{1}{T}\int_0^T M_\varphi^{-1} F_1\big(t, x(t,\varphi)\big)dt. \tag{13}$$

If there exists $\alpha \in W$ with $\mathcal{F}(\alpha) = 0$ and $det\big(D_\varphi \mathcal{F}(\alpha)\big) \neq 0$, then for system (10) there is a T-periodic solution $x(t,\epsilon)$ such that $x(t,\epsilon) \longrightarrow \alpha$ as $\epsilon \longrightarrow 0$.
For a proof of the above result, see [37].

3.2 Periodic Solutions in a Zero-Hopf Bifurcation of the 4DLVS

The proposition below shows that the existence of the 4DLVS parameter such that any point on the line of singularities (8) is a zero-Hopf equilibrium point.
Proposition (3.2.1): If the following condition is satisfied, then system (2) has a double zero-Hopf equilibrium points with conditions (5) and (6) at the line of singularities (8):

$$a_{21} = \frac{a_{41}(\omega^2 + a_{32}^2)}{\omega^2 + a_{32}a_{41}}. \tag{14}$$

Proof: At any point localized at the line of singularities (8), the characteristic polynomial $P(\lambda)$ of the linearization of system (7) is given by

$$P(\lambda) = \lambda^4 - S_1\lambda^3 + S_2\lambda^2 - S_3\lambda, \tag{15}$$

where

$$\begin{aligned} S_1 &= \frac{1}{a_{21} - a_{41}}(a_{41} - a_{31})(a_{21} + a_{31} - a_{32} - a_{41}) + (a_{31} - a_{41} - a_{43})t, \\ S_2 &= \frac{a_{41}}{a_{21} - a_{41}}(a_{31} - a_{32} - a_{41})(a_{21} + a_{31} - a_{32} - a_{41}), \\ S_3 &= \frac{a_{41}}{(a_{21} - a_{41})^2}((a_{31} - a_{32} - a_{41})(a_{21} + a_{31} - a_{32} - a_{41})^2(a_{31} - a_{41}) \\ &\quad -(a_{31} - a_{41} - a_{43})(a_{21} - a_{41})t). \end{aligned} \tag{16}$$

Assume that at each point along the line of singularity (8), system (7) has two zeros and a pair of purely imaginary complex eigenvalues. Hence, $P(\lambda)$ must take the following form.

$$P(\lambda) = \lambda^2(\lambda^2 + \omega^2),$$

where $\omega > 0$. The proof is directly made by comparing the coefficients in both $P(\lambda)$. After doing computations using the Maple software, the condition (14) will be found. Conversely, the Jacobian matrix of system (7) under the family of condition (14) at the line of singularity (8) has the double zero and a pair of purely conjugate complex eigenvalues $\pm i\omega$ will not depend on the value of t. This means that all points along the line of singularity (8) are therefore the zero-Hopf equilibrium points.

Theorem (3.2.2): Consider system (2) with conditions (5), (6) and (14) in Propositions (2.2), (2.3) and (3.2.1), respectively, are satisfied. Let $a_{21} = \frac{a_{41}(\omega^2 + a_{32}^2)}{\omega^2 + a_{32}a_{41}} + \epsilon\mu_1$, where ϵ is too small positive parameter and $\omega > 0$. Using the first-order averaging method, we are unable to find any periodic solution bifurcating from the zero-Hopf equilibrium point satisfing condition (14) in Proposition (3.2.1).

Proof: Suppose that the perturbation $a_{21} = \frac{a_{41}(\omega^2+a_{32}^2)}{\omega^2+a_{32}a_{41}} + \epsilon\mu_1$ holds. Firstly, after we change the line of singularities' equilibrium point to the origin, the four-dimensional Lotka-Volterra system satisfing the above condition is expressed in the following

$$\begin{cases} \dot{x} = x(a_{31}x - \frac{\ell_2}{\ell_0}y + \frac{a_{31}a_{32}\ell_0}{\ell_1}) - \frac{a_{32}\ell_2}{\ell_1}y, \\ \dot{y} = y(\frac{\ell_3}{\ell_0}x + a_{32}y - \frac{a_{31}a_{32}\ell_0}{\ell_1}) - \frac{a_{31}\ell_3}{\ell_1}x, \\ \dot{z} = z(a_{31}x + a_{32}y - a_{43}w), \\ \dot{w} = w(a_{31}x + a_{32}y + a_{43}z). \end{cases} \tag{17}$$

By rescaling the variables $(x, y, z, w) = (\epsilon x, \epsilon y, \epsilon z, \epsilon w)$, system (17) becomes

$$\begin{cases} \dot{x} = \epsilon x(a_{31}x - \frac{\ell_2}{\ell_0}y) - \frac{\ell_2 a_{32}}{\ell_1}y + \frac{\ell_0 a_{32}a_{31}}{\ell_1}, \\ \dot{y} = \epsilon y(\frac{\ell_3}{\ell_0}x + a_{32}y) - \frac{a_{31}\ell_3}{\ell_1}x - \frac{a_{31}a_{32}\ell_0}{\ell_1}, \\ \dot{z} = \epsilon z(a_{31}x + a_{32}y - a_{43}w), \\ \dot{w} = \epsilon w(a_{31}x + a_{32}y + a_{43}z), \end{cases} \tag{18}$$

where $\ell_0 = \omega^2 + a_{31}a_{32}, \ell_1 = \epsilon\mu_1\ell_0 - a_{31}a_{32}(a_{31} - a_{32}), \ell_2 = \epsilon\mu_1\ell_0 - a_{32}(\omega^2 + a_{31}^2)$ and $\ell_3 = \epsilon\mu_1\ell_0 + a_{31}(\omega^2 + a_{32}^2)$. Now, the linearized system (18) at the origin is not in the real Jordan form, when $\epsilon = 0$, i.e. as

$$\begin{pmatrix} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For performing that, we use the following linear change of coordinates

$$(x, y, z, w) = P(X, Y, Z, W), \quad P = \begin{pmatrix} p_1 & p_2 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -a_{31}(\omega^2 + a_{32}^2) \\ 0 & 0 & \dfrac{1}{\omega a_{32}(a_{31} - a_{32})} & -1 \end{pmatrix},$$

where $p_1 = \frac{a_{32}(a_{32}(a_{31}-\omega)+\omega(\omega+a_{31}))}{a_{31}(\omega^2+a_{32}^2)}$ and $p_2 = \frac{a_{32}(\omega(a_{31}-\omega)-a_{32}(\omega+a_{31}))}{a_{31}(\omega^2+a_{32}^2)}$.

System (18) in the new variables (X, Y, Z, W) becomes

$$\begin{cases} \dot{X} = -\omega Y + \frac{\epsilon}{\delta_1}\left(\delta_3 X^2 + (\delta_9 Y - \delta_8)X - \delta_3 Y^2 + \frac{a_{32}-\omega}{a_{32}+\omega}\delta_8 Y\right) + O(\epsilon^2), \\ \dot{Y} = \omega X + \frac{\epsilon}{\delta_1}\left(\delta_2 X^2 + \left(\delta_9 Y - \frac{a_{32}+\omega}{a_{32}-\omega}\delta_8\right)X - \delta_2 Y^2 + \delta_8 Y\right) + O(\epsilon^2), \\ \dot{Z} = \frac{\epsilon\,\delta_7}{\omega^2+a_{32}^2}(\omega a_{32}(a_{31} - a_{32})W^2 - ZW + \frac{\delta_4}{\delta_7}(X + \frac{(\omega-a_{32})}{(a_{32}+\omega)}Y)Z) + O(\epsilon^2), \\ \dot{W} = \frac{\epsilon}{\delta_5}(a_{43}\delta_5 W^2 + (\delta_6 X + \frac{\delta_6(\omega-a_{32})}{a_{32}+\omega}Y - a_{43}(\omega^2 + a_{32}^2)Z)W) + O(\epsilon^2), \end{cases} \tag{19}$$

where

$\delta_1 = 2\omega a_{31}a_{32}(a_{31} - a_{32})(\omega^2 + a_{32}^2)(\omega^2 + a_{31}a_{32}),$

$\delta_2 = \omega a_{31}a_{32}^2(a_{32} - \omega)(a_{31} - a_{32})^2(a_{32}(a_{31} - \omega) + \omega(\omega + a_{31})),$

$\delta_3 = \omega a_{31}a_{32}^2(a_{31} - a_{32})^2(a_{32} + \omega)(a_{32}(\omega + a_{31}) + \omega(\omega - a_{31})),$

$\delta_4 = a_{32}(a_{31} - a_{32})(a_{32} + \omega), \quad \delta_5 = \omega a_{32}(a_{31} - a_{32})(\omega^2 + a_{32}^2),$
$\delta_6 = \omega a_{32}^2(a_{31} - a_{32})^2(a_{32} + \omega), \quad \delta_7 = a_{43}(\omega^2 + a_{32}^2)(1 + a_{31}(\omega^2 + a_{32}^2)),$
$\delta_8 = \mu_1(\omega - a_{32})(a_{32} + \omega)(\omega^2 + a_{31}a_{32})^3, \ \ \delta_9 = 4a_{31}\omega^2 a_{32}^2(a_{31} - a_{32})^2(\omega^2 + a_{31}a_{32}).$
System (19) is in the form

$$\begin{cases} \dot{X} = -\omega Y + \epsilon\, G_1(X,Y,Z,W), \\ \dot{Y} = \omega X + \epsilon\, G_2(X,Y,Z,W), \\ \dot{Z} = 0 + \epsilon\, G_3(X,Y,Z,W), \\ \dot{W} = 0 + \epsilon\, G_4(X,Y,Z,W). \end{cases} \tag{20}$$

Note that system (20) is expressed as of the form (10). We can use the first-order averaging method described in Theorem 3.1.1, since it has a normal form. We observe that the system (20) is equivalent to system (10) by taking

$$x = \begin{pmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \\ \mathrm{W} \end{pmatrix}, \ \ F_0(t,x) = \begin{pmatrix} -\omega Y \\ \omega X \\ 0 \\ 0 \end{pmatrix}, \ \text{and} \ \ F_1(t,x) = \begin{pmatrix} G_1(X,Y,Z,W) \\ G_2(X,Y,Z,W) \\ G_3(X,Y,Z,W) \\ G_4(X,Y,Z,W) \end{pmatrix}.$$

The unperturbed system(20) must first be solved. The solution $x(t,\varphi) = (X(t),Y(t),Z(t),W(t))$ in the following unperturbed part

$$\begin{cases} \dot{X} = -\omega Z, \\ \dot{Y} = \omega X, \\ \dot{Z} = 0, \\ \dot{W} = 0, \end{cases} \tag{21}$$

which satisfies the initial condition $(X(0),Y(0),Z(0),W(0)) = (X_0,Y_0,Z_0,W_0) \in \mathbb{R}^4$ is represented as

$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \\ W(t) \end{pmatrix} = \begin{pmatrix} X_0\sin(\omega t) + Y_0\cos(\omega t) \\ Y_0\sin(\omega t) - X_0\cos(\omega t) \\ Z_0 \\ W_0 \end{pmatrix},$$

when $(X_0,Y_0,Z_0,W_0) \neq (0,0,0,0)$, these solutions are periodic of period $\frac{2\pi}{\omega}$. Therefore, we can use Theorem (3.1.1) because the unperturbed system (21) of (19) is isochronous. The fundamental matrix solution $M_\varphi(t)$ and its inverse $M_\varphi^{-1}(t)$ of system (21) are obtained by

$$M_\varphi(t) = \begin{pmatrix} \sin(\omega t) & \cos(\omega t) & 0 & 0 \\ -\cos(\omega t) & \sin(\omega t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ \text{and} \ \ M_\varphi^{-1}(t) = \begin{pmatrix} \sin(\omega t) & -\cos(\omega t) & 0 & 0 \\ \cos(\omega t) & \sin(\omega t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The averaged function (13) is obtained by

$$\mathcal{F}(\varphi) = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} M_\varphi(t)^{-1} F_1(t,x(t,\varphi))dt = \begin{pmatrix} F_{11}(X_0,Y_0,Z_0,W_0) \\ F_{21}(X_0,Y_0,Z_0,W_0) \\ F_{31}(X_0,Y_0,Z_0,W_0) \\ F_{41}(X_0,Y_0,Z_0,W_0) \end{pmatrix},$$

where

$$F_{11}(\varphi) = \frac{\mu_1(a_{32}(X_0 + Y_0) + \omega(X_0 - Y_0))(\omega^2 + a_{31}a_{32})^2}{2a_{32}a_{31}(a_{31} - a_{32})(\omega^2 + a_{32}^2)},$$

$$F_{21}(\varphi) = \frac{\mu_1(\omega(X_0 + Y_0) - a_{32}(X_0 - Y_0))(\omega^2 + a_{31}a_{32})^2}{2a_{32}a_{31}(a_{31} - a_{32})(\omega^2 + a_{32}^2)}, \tag{22}$$

$$F_{31}(\varphi) = a_{43}(\omega a_{32}(a_{31} - a_{32})W_0 - Z_0)(1 + a_{31}(\omega^2 + a_{32}^2))W_0,$$
$$F_{41}(\varphi) = \frac{a_{43}(\omega a_{32}(a_{31} - a_{32})W_0 - Z_0)W_0}{\omega a_{32}(a_{31} - a_{32})}.$$

Then, system (22) has the following solutions

$$s_1 = (0,0,0,0), \qquad s_2 = (0,0, \omega a_{32}(a_{31} - a_{32})W_0, 0),$$

In addition, the determinants of the Jacobian matrix at s_1 and s_2 take the values

$$\det\left(\frac{\partial\mathcal{F}(\varphi)}{\partial x}\right)\Big|_{s_1} = 0, \quad \text{and} \quad \det\left(\frac{\partial\mathcal{F}(\varphi)}{\partial x}\right)\Big|_{s_2} = 0.$$

This means that s_1 and s_2 are not acceptable solutions. As a result, the averaging technique explained in Theorem (3.1.1) does not give any information about the possible periodic solutions bifurcating from the zero-Hopf equilibrium point.

4. Conclusion

In this article, we have considered the four-dimensional Lotka-Volterra systems (4DLV) system. We have then found three first integrals of this type of system, this means that the 4DLV system is integrable under the suitable conditions. Also, using these conditions, we have found that the system has a line of singularities for which a zero-Hopf equilibrium occurs at each point localized at that line of singularities. Moreover, only one parameter of the system exists that displays such equilibrium point. The first-order averaging technique is then used to find a periodic solution, but the first-order averaging theory described in Theorem (3.1.1) does not provide any information about the possible periodic orbits bifurcating from the zero-Hopf equilibrium point.

References

[1] T. Idema, "The behaviour and attractiveness of the Lotka-Volterra equations," *Phd, Universiteit Leiden,* 2005.

[2] R. H. Salih, Hopf bifurcation and centre bifurcation in three dimensional Lotka-Volterra systems, 2015.

[3] J. D. Murray, "Mathematical biology II: Spatial models and biomedical applications," *Monographs on Applied and Computational Mathematics,* vol. 3, 1973.

[4] R. Wang and . D. Xia, "Bifurcations and chaotic dynamics in a 4-dimensional competitive Lotka--Volterra system," *Nonlinear Dynamics,* vol. 59, no. 3, pp. 411-422, 2010.

[5] H. Lyu and P. G. Jablonski, "Four-dimensional discrete-time Lotka-Volterra models with an application to ecology," *arXiv preprint arXiv:1211.5861,* 2012.

[6] H. Kowgier, "On Four--Dimensional Lotka--Volterra Models," *Polish Journal of Environmental Studies,* vol. 18, no. 3B, 2009.

[7] G. Farhan, T. A. Balasim and S. Al-Nassir, "On the Stability of Four Dimensional Lotka-Volterra Prey-Predator System," *Iraqi Journal of Science,* vol. 64, no. 8, pp. 5009-5030, 2023.

[8] A. Antonov, S. Nenov and . T. Tzvetkov, "prey-predator tridiagonal 4-dimensional models," *Communications in Applied Analysis,* vol. 23, no. 1, pp. 51-60, 2019.

[9] A. Ferragut and C. Valls, "On the Complete Integrability of the Raychaudhuri Differential System in R 4 and of a CRNT Model in R 5," *Qualitative Theory of Dynamical Systems,* vol. 17, no. 1, pp. 291-307, 2018.

[10] J. Llibre, A. Murza and A. Teruel, "First integrals of a class of $ n $-dimensional Lotka-Volterra differential systems," *arXiv preprint arXiv:1707.08854,* 2017.

[11] J. Llibre and C. Valls, "Darboux Integrability of a Simplified Friedman--Robertson--Walker Hamiltonian System," *Journal of Nonlinear Mathematical Physics,* vol. 19, no. 4, pp. 607-617, 2012.

[12] J. Llibre and C. Valls, "On the Darboux integrability of the logarithmic galactic potentials," *Journal of Geometry and Physics,* vol. 121, pp. 279-287, 2017.

[13] J. Guckenheimer, On a codimension two bifurcation, Springer, 2006, pp. 99-142.

[14] J. Guckenheimer and P. Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields, vol. 42, 2013.

[15] Y. A. Kuznetsov, Elements of applied bifurcation theory, vol. 112, Springer, 1998.

[16] H. Maoan, "Existence of periodic orbits and invariant tori in codimension two bifurcations of three dimensional systems," *Journal of Systems Science and Mathematical Sciences,* vol. 18, no. 4, p. 403, 1998.

[17] J. Scheurle and J. Marsden, "Bifurcation to quasi-periodic tori in the interaction of steady state and Hopf bifurcations," *SIAM journal on mathematical analysis,* vol. 15, no. 6, pp. 1055-1074, 1984.

[18] I. Baldoma and T. M. Seara, "Breakdown of heteroclinic orbits for some analytic unfoldings of the Hopf-zero singularity," *Journal of Nonlinear Science,* vol. 16, pp. 543-582, 2006.

[19] I. Baldoma and T. M. Seara, "The inner equation for generic analytic unfoldings of the Hopf-zero singularity," *Discrete Contin. Dyn. Syst. Ser. B,* vol. 10, no. 2-3, pp. 323-347, 2008.

[20] H. W. Broer and G. Vegter, "Subordinate Sil'nikov bifurcations near some singularities of vector fields having low codimension," *Ergodic Theory and Dynamical Systems,* vol. 4, no. 4, pp. 509-525, 1984.

[21] A. R. Champneys and V. Kirk, "The entwined wiggling of homoclinic curves emerging from saddle-node/Hopf instabilities," *Physica D: Nonlinear Phenomena,* vol. 195, no. 1-2, pp. 77-105, 2004.

[22] V. Castellanos, J. Llibre and I. Quilantan, "Simultaneous periodic orbits bifurcating from two zero-Hopf equilibria in a tritrophic food chain model," *Journal of Applied Mathematics and Physics,* vol. 2013, 2013.

[23] I. A. Garcia, J. Llibre and S. Maza, "On the periodic orbit bifurcating from a zero Hopf bifurcation in systems with two slow and one fast variables," *Applied Mathematics and Computation,* vol. 232, pp. 84-90, 2014.

[24] J. Llibre, "Periodic orbits in the zero-Hopf bifurcation of the Rossler system," *Romanian Astron. J,* vol. 24, no. 1, pp. 49-60, 2014.

[25] J. Llibre, R. D. Oliveira and C. Valls, "On the integrability and the zero-Hopf bifurcation of a Chen-Wang differential system," *Nonlinear Dynamics,* vol. 80, pp. 353-361, 2015.

[26] J. Llibre and Y. Tian, "The zero-Hopf bifurcations of a four-dimensional hyperchaotic system," *Journal of Mathematical Physics,* vol. 62, no. 5, 2021.

[27] A. Feddaoui, J. Llibre and A. Makhlouf, "4-dimensional zero-Hopf bifurcation for polynomial differentials systems with cubic homogeneous nonlinearities via averaging theory," *International Journal of Dynamical Systems and Differential Equations,* vol. 10, no. 4, pp. 321-328, 2020.

[28] M. Amar, B. Lilia and others, "Periodic Solutions of Some Polynomial Differential Systems in R 4," *Journal of Applied Mathematics and Physics,* vol. 5, no. 1, p. 194, 2017.

[29] D. Djedid, E. O. Bendib and A. Makhlouf, "Four-dimensional Zero-Hopf Bifurcation of Quadratic Polynomial Differential System, via Averaging Theory of Third Order," *Journal of Dynamical and Control Systems,* pp. 1-16, 2021.

[30] A. Feddaoui, J. Llibre, C. Berhail and A. Makhlouf, "Periodic solutions for differential systems in ℝ 3 and ℝ 4," *Applied Mathematics and Nonlinear Sciences,* vol. 6, no. 1, pp. 373-380, 2021.

[31] W. Wu and Z. Chen, "Hopf bifurcation and intermittent transition to hyperchaos in a novel strong four-dimensional hyperchaotic system," *Nonlinear Dynamics,* vol. 60, pp. 615-630, 2010.

[32] L. Zhou, Z. Zhao and F. Chen, "Stability and Hopf bifurcation analysis of a new four-dimensional hyper-chaotic system," *Modern Physics Letters B,* vol. 34, no. 29, p. 2050327, 2020.

[33] L. Barreira, J. Llibre and C. Valls, "Integrability and zero-Hopf bifurcation in the Sprott A system," *Bulletin des Sciences Math{\'e}matiques,* vol. 162, p. 102874, 2020.

[34] J. A. Sanders, F. Verhulst and J. Murdock, Averaging methods in nonlinear dynamical systems, vol. 59, Springer, 2007.

[35] J. E. Marsden and M. McCracken, The Hopf bifurcation and its applications, vol. 19, Springer Science \& Business Media, 2012.

[36] S. N. Chow and J. K. Hale, Methods of bifurcation theory, vol. 251, Springer Science \& Business Media, 2012.

[37] A. Buica, J. P. Francoise and J. Llibre, "Periodic solutions of nonlinear periodic differential systems with a small parameter," *Communications on Pure and Applied Analysis*, vol. 6, no. 1, p. 103, 2007.