*DOI: 10.24996/ijs.2024.65.10.26*

**ISSN: 0067-2904**

# Centralizers and Double Centralizers for Prime and Semiprime Γ-rings

**Aya Hussein Khuder*, Abdulrahman H. Majeed**
*Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq*

Received: 4/5/2023 Accepted: 8/8/2023 Published: 30/**10**/2024

**Abstract**
The purpose of this paper is to discuss the centralizers and the double centralizers in prime and semiprime Γ −rings with fulfilling certain identities.

**Keywords:** Centralizers, Double Centralizers, Semiprime Γ −Rings.

## التمركزات والتمركزات الثنائية لحلقات− Γ الاولية وشبه الاولية

**اية حسين خضير*, عبدالرحمن حميد مجيد**
قسم الرياضيات, كلية العلوم, جامعة بغداد, بغداد, العراق

**الخلاصه**
الغرض من هذا العمل مناقشة التمركزات والتمركزات الثنائية لحلقات− Γ الاولية وشبه الاولية مع تحقيق تطبيقات معينة.

## 1. Introduction

The definition of Γ-ring was introduced by Barnes [1]. Let $K$ and Γ be two abelian groups. If there is a mapping $(a\,\alpha\,b) \to (a\,\alpha\,b)$ of $K \times \Gamma \times K \to K$, fulfilling the following, for any $a, b, c \in K$ and $\alpha, \beta \in \Gamma$.

i. $(a+b)\,\alpha\,c = a\,\alpha\,c + b\,\alpha\,c$,
$a\,(\alpha+\beta)\,b = a\,\alpha\,b + a\,\beta\,b$,
$a\,\alpha\,(b+c) = a\,\alpha\,b + a\,\alpha\,c$,
ii. $(a\,\alpha\,b)\,\beta\,c = a\,\alpha\,(b\,\beta\,c)$,
then $K$ is named a $\Gamma$-ring.

Every ring $K$ is a $\Gamma$-ring with $K=\Gamma$. A Γ-ring not necessary be a ring. The concept of Gamma ring is a generalization of rings, where proposed by Nobuswa [2]. Barnes [1] diminished slightly the requirements in the definition of Γ-ring as in Nobuswa.

In [3], D. Özden, M.A. Özturk and Y. B. Jun, defined a $\Gamma$ −subring. A Γ-subring of $\Gamma$-ring $K$ is an additive subgroup $S$ of $K$ such that $S\,\Gamma\,S \subseteq S$. Let $K$ be a $\Gamma$-ring, then $K$ is named a commutative $\Gamma$-ring if $a\,\alpha\,b = b\,\alpha\,a$, holds for any $a, b \in K$ and $\alpha \in \Gamma$ [4]. A subset $U$ of a $\Gamma$-ring $K$ is called a right (resp. left) ideal of $K$ if $U$ is an additive subgroup of $K$ and $U\,\Gamma\,K = \{a\,\alpha\,x \colon a \in U, \alpha \in \Gamma, x \in K\}$ (resp. $K\Gamma\,U = \{x\,\alpha\,a \colon a \in U, \alpha \in \Gamma, x \in K\}$ ) is

*Email: alaa.w@sc.uobaghdad.edu.iq

contained in $U$. If $U$ is both a left and a right ideal, then $U$ is called a two-sided ideal, or simply is an ideal of $K$ [5].

A $\Gamma$-ring $K$ is named prime $\Gamma$-ring if $a\,\Gamma\,K\,\Gamma b = 0,$ implies $a$=0 or $b$=0, where $a, b \in K$. A $\Gamma$-ring $K$ is named semiprime ring if $a\,\Gamma\,K\,\Gamma a = 0,$ implies $a$=0, where $a \in K$ [6].

Let $K$ be a $\Gamma$-ring, then $K$ is named $n-$torsion free if $n\,a = 0$, yields $a = 0$, for every $a \in K$, where n is positive integer [7].

Let $K$ be a $\Gamma$-semiring, an element $1 \in K$, is named unity if for any $x \in K$ there exists $\alpha \in \Gamma$ such that $x\,\alpha\,1 = 1\,\alpha\,x = x$ [8].

In [9], Ceven and Uzturk defined the derivation and Jordan derivation in Γ-rings. Let K be a Γ-ring and $d : K \rightarrow K$ an additive map. Then $d$ is named a derivation (resp. Jordan derivation), if $d\,(x\,\alpha\,y) = d\,(x)\,\alpha\,y + x\,\alpha\,d\,(y)$ (resp. $d\,(x\,\alpha\,x) = d\,(x)\,\alpha\,x + x\,\alpha\,d\,(x)$), for any $x$, y ∈ K and α ∈ Γ. Every derivation of $K$ is Jordan derivation but the opposite in general is need not to be true ( see [9]).

Let $K$ be a $\Gamma$-ring with center $Z(K)$, a mapping $d$ from $K$ into itself is named $\Gamma$-centralizing on a subset $S$ of $K$ if $[x, d\,(x)]_\alpha \in Z(K)$ for every $x \in S$ and $\alpha \in \Gamma$, in the special case when $[x, d\,(x)]_\alpha = 0$ hold for any x ∈ $S$ and α ∈ $\Gamma$, the mapping $d$ is named $\Gamma$-commuting on $S$ [7]. Many researchers have studied centralizers and derivations in prime and semiprime $\Gamma$- rings, see [10-18]. The purpose of this paper is to discuss centralizers and double centralizers in semiprime $\Gamma-$rings with fulfilling certain identities.

## 2. Basic Concepts

We begin our discussion with the following definitions and lemmas which are useful for the proof of our main results.

**Definition 2.1[15]**

Let K be a Γ-ring, $d$ be called inner derivation of $K$, if there exists $a \in K$, such that $d(x) = [a, x]_\alpha$ for all $x \in$ K and $\alpha \in \Gamma$.

**Definition 2.2 [16]**

Let $K$ be a Γ-ring, for any $x, y \in K$ and α ∈ Γ, the symbol $[x, y]_\alpha = x\,\alpha\,y - y\,\alpha\,x$, is denoted to the commutator, and $(x \circ y)_\alpha = x\,\alpha\,y + y\,\alpha\,x$.

**Lemma 2.3 [16]**

If $K$ is a $\Gamma$-ring, then the following are hold for any $a, b, c \in K$ and $\alpha, \beta \in \Gamma$:

i. $[a, b]_\alpha + [b, a]_\alpha = 0.$
ii. $[a + b, c]_\alpha = [a, c]_\alpha + [b, c]_\alpha.$
iii. $[a, b + c]_\alpha = [a, b]_\alpha + [a, c]_\alpha.$
iv. $[a, b]_{\alpha+\beta} = [a, b]_\alpha + [a, b]_\beta.$
v. $[a\,\beta\,b, c]_\alpha = a\,\beta[b, c]_\alpha + [a, c]_\alpha\,\beta\,b + a\,\beta\,c\,\alpha\,b - a\,\alpha\,c\,\beta\,b$ .

**Definition 2.4[17]**

Let $K$ be a $\Gamma$-ring. An additive mapping is called a left (resp. right) centralizer $T: K \rightarrow K$ if $T(x\alpha\,y) = T\,(x)\,\alpha\,y$ $(resp.\, T\,(x\,\alpha\,y) = x\,\alpha\,T\,(y))$ holds for any $x, y \in K$ and α ∈ Γ. A centralizer is both a left and right centralizer.

**Example 2.5**

Let F be a field, and $D_2$ (F) be a Γ-ring of all diagonal matrices of degree 2, where $\Gamma = \{ \begin{bmatrix} 0 & 0 \\ 0 & n \end{bmatrix} \mid n \in F \}$. Define $T: D_2(F) \rightarrow D_2(F)$ by:

$$T\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \text{ for any } a, b \in F .$$

Then T is a centralizer.

**Definition 2.6[17]**

Let $K$ be a $\Gamma$-ring. An additive mapping $T: K \rightarrow K$ is called Jordan left (resp. right) centralizer if $T(x\alpha x) = T(x)\alpha x$ ( resp. $T(x\alpha x) = x\alpha T(x)$, for any $x \in K$ and $\alpha \in \Gamma$.

**Remark 2.7**

Every centralizer is Jordan centralizer but the converse in general is need not to be true, as the following example shows:

**Example 2.8**

Let $F$ be a field, and K be a $\Gamma$-ring of all matrices of the from:

$$x = \begin{bmatrix} 0 & a & c & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix} , \text{ for any } a, b, c \in F,$$

and

$$\Gamma = \{ \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, n \text{ is integer}\}$$

Let $T: K \rightarrow K$ be an additive mapping defined as:

$$T(x) = \begin{bmatrix} 0 & a & c & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ for any } a, c \in F, x \in K.$$

Then $T$ is a Jordan centralizer but not centralizer.

**Theorem 2.9** [17]

Let $K$ be a 2-torsion free semiprime Γ-ring, then every left (resp. right) Jordan centralizer is a left (resp. right) centralizer.

**Definition 2.10**

Let $d$ and $g$ be additive mappings on $\Gamma$-ring $K$, a pair $(d, g)$ is called a derivation pair if the following equations hold:

$$d(x \alpha y \beta x) = d(x) \alpha y \beta x + x \alpha g(y) \beta x + x \alpha y \beta d(x)$$
$$\text{for any } x, y \in K, \alpha, \beta \in \Gamma.$$

$$g(x \alpha y \beta x) = g(x) \alpha y \beta x + x \alpha d(y) \beta x + x \alpha y \beta g(x)$$
$$\text{for any } x, y \in K, \alpha, \beta \in \Gamma.$$

## 3. Main results

**Lemma 3.1**

Let $K$ be a prime $\Gamma$-ring, and $U$ be a non-zero ideal of $K$. Let $T: K \rightarrow K$, be a left centralizer of $K$. If $T = 0$ on $U$, then $T = 0$ on $K$.

**Proof:** We have

$$T(x) = 0 \quad \text{for all } x \in U. \tag{1}$$

Replacing $x$ by $r\,\alpha\,x$ in (1), where $r \in K$

$$T(r\alpha x) = 0 \text{ for all } x \in U, r \in K, \alpha \in \Gamma.$$

Since $T$ is left centralizer, we have

$$T(r)\alpha x = 0 \text{ for all } r \in K, x \in U, \alpha \in \Gamma. \tag{2}$$

Replace $x$ with $s\,\beta\,x$ in (2), we get

$$T(r)\alpha s\,\beta\,x = 0 \quad for\ all\ r, s \in K, x \in U, \alpha, \beta \in \Gamma, \tag{3}$$

hence

$$T(r)\alpha K\,\beta\,x = 0 \quad for\ all\ r \in K, x \in U, \alpha, \beta \in \Gamma$$

by the primness of $K$, and $U$ be a non-zero ideal of $K$, we have

$$T(r) = 0 \quad for\ all\ r \in K.$$

**Theorem 3.2**

Let $K$ be a non-commutative prime Γ-ring, let $U$ be a non-zero ideal of $K$, and $T: K \to K$ be a left centralizer. If $T(x) \in Z(K)$, holds for any $x \in U$, then $T = 0$.

**Proof:**

Since

$$[T(x), r]_\alpha = 0 \quad \text{for all } r \in K, x \in U \text{ and } \alpha \in \Gamma \tag{1}$$

Putting $x\,\beta\,z$ for $x$ in (1), where $z \in K$, and $\alpha, \beta \in \Gamma$, we get

$$[T(x), r]_\alpha\,\beta\,z + T(x)\,\beta\,[z, r]_\alpha = 0 \quad \text{for all } r, z \in K, x \in U \text{ and } \alpha, \beta \in \Gamma. \tag{2}$$

Hence,

$$T(x)\,\beta\,[z, r]_\alpha = 0, \text{for all } r, z \in K, x \in U \text{and } \alpha, \beta \in \Gamma. \tag{3}$$

By replacing $x$ with $x\,\sigma\,w$, in (3), where $w \in K$, $\sigma \in \Gamma$, give

$$T(x)\,\sigma\,w\,\beta\,[z, r]_\alpha = 0 \quad \text{for all } r, z, w \in K, x \in U \text{ and } \alpha, \beta, \sigma \in \Gamma. \tag{4}$$

By the primness and non-commutative of $K$, follows $T(x) = 0$, for any $x \in U$, using Lemma 3.1, we have $T = 0$.

**Theorem 3.3**

Let $K$ be a semiprime Γ-ring, $U$ be an ideal of $K$, and let $T: K \to K$ be a centralizer of $K$. If $T(x)\,\alpha\,T(y) = 0$, for any $x, y \in U$, then $T = 0$ on $U$. In case $K$ is a prime Γ-ring, then $T = 0$.

**Proof:** We have

$$T(x)\alpha\,T(y) = 0 \quad \text{for all } x, y \in U\ and\ \alpha \in \Gamma$$

Replace y by $r\,\beta\,x$ in the above relation, since $T$ is centralizer, we get

$$T(x)\alpha\,K\,\beta\,T(x) = 0 \quad \text{for all } x \in U,\ \alpha, \beta \in \Gamma.$$

By the semiprimness of $K$, we get

$T(x) = 0$ for all $x \in U$.

In case $K$ is prime $\Gamma$-ring and using Lemma 3.1, which complete the proof.

**Theorem 3.4**

Let $K$ be a 2-torsin free semiprime $\Gamma$-ring, and let $T$: $K \to K$ be a left centralizer of $K$, such that $T(x \circ y)_\alpha = 0$ and $y\alpha x\beta z = y\beta x\alpha z$, for any $x, y \in K$, $\alpha, \beta \in \Gamma$ then $T(x)=0$.

**Proof:**

We have

$$T(x \circ y)_\alpha = T(x\,\alpha\,y + y\,\alpha\,x) = 0 \quad \text{for all } x, y \in K, \alpha \in \Gamma \tag{1}$$

Gives us

$$T(x)\alpha\,y + T(y)\alpha\,x = 0 \quad \text{for all } x, y \in K, \alpha \in \Gamma. \tag{2}$$

Replace y by $y\,\beta\,z + z\,\beta\,y$ in (2), we obtain

$$T(x)\,\alpha\,(y\,\beta\,z+z\,\beta\,y)+T(y\,\beta\,z+z\,\beta\,y)\,\alpha\,x=0 \quad \text{for any } x, y, z \in K, \alpha, \beta \in \Gamma.$$

Now, from (1), we get

$$T(x)\alpha\,(y\,\beta\,z+z\,\beta\,y)\;=0 \quad for\ all\ x,y,z \in K, \alpha,\beta \in \Gamma. \qquad (3)$$

Replace z with $y\,\gamma\,z + z\,\gamma\,y$ in (3), we get

$$2T(x)\alpha\,(y\,\gamma\,z\,\beta\,y)=0 \quad for\ all\ x,y,z \in K, \alpha,\beta,\gamma \in \Gamma. \qquad (4)$$

Since $K$ is a 2-torsion free, then (4) leads to

$$T(x)\alpha\,(y\,\gamma\,z\beta\,y)=0 \quad for\ all\ x,y,z \in K, \alpha,\ \beta,\gamma \in \Gamma. \qquad (5)$$

Replace z by $z\,\sigma T(x)$ in (5), we get

$$T(x)\alpha\,y\,\gamma\,z\;\;\sigma\,T(x)\,\alpha\,y=0 \quad for\ all\ x,y,z \in K, and\ \alpha,\gamma,\sigma \in \Gamma. \qquad (6)$$

By the semiprimness of $K$, we get $T=0$.

**Theorem 3.5**

Let $K$ be a prime $\Gamma$ -ring, let $U$ be a non-zero ideal of $K$, and $T$: $K\rightarrow K$, be an additive mapping which satisfies $T(r\,\alpha\,x)=T(r)\alpha\,x$, for any $r \in K$, $x \in U$, $\alpha \in \Gamma$. Then $T$ is a left centralizer of $K$.

**Proof:**

By the assumption, we have

$$T(r\,\alpha\,x)=T(r)\alpha\,x \quad \text{for all } r \in K, x \in U, \alpha \in \Gamma$$

Replace x by $s\,\beta\,x$ in the above relation, we get

$$T(r\,\alpha\,s\,\beta\,x)=T(r\,\alpha\,s)\,\beta\,x=T(r)\alpha\,s\,\beta\,x \quad for\ all\ r,s \in K, x \in U, \alpha,\beta \in \Gamma$$

i.e.,

$$(T(r\,\alpha\,s)-T(r)\alpha\,s\;)\beta\,t\,\gamma\,x=0 \quad for\ all\ r,s,t \in K, x \in U, \alpha,\beta \in \Gamma$$

By the primness of $K$ and $U$ is a non-zero, we get

$$T(r\,\alpha\,s)=T(r)\alpha\,s \quad \text{for all } r,s \in K, \alpha \in \Gamma.$$

Then $T$ is a left centralizer of $K$.

**Definition 3.6**

Let $K$ be $\Gamma$ -ring, let $T$, $S$: $K\rightarrow K$, be additive mappings, then a pair $(T,S)$ is named a double centralizer, if $T$ is a left centralizer, $S$ is a right centralizer, and satisfy a balanced requirement $x\,\alpha\,T(y)=S(x)\,\alpha\,y$ , for any $x, y \in K$.

**Example 3.7**

Let $F$ be a field, and $K_2(F)$ be a Γ-ring of all 2 by 2 matrices with usual addition and multiplication, and $\Gamma=\{\begin{bmatrix} n & 0 \\ 0 & 0 \end{bmatrix}$, $n$ is integer},

Define $T, S$: $K_2(F) \rightarrow K_2(F)$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)=\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix},\ \textit{for any } a, b, c, d \in F.$$

$$S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)=\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix},\ \textit{for any } a, b, c, d \in F.$$

It is clear that $T$ is a left centralizer and $S$ is a right centralizer satisfy the condition

$$x\,\alpha\,T(y)=S(x)\,\alpha\,y.$$

Therefore, $(T, S)$ is a double centralizer.

**Remark 3.8**

Let $K$ be a Γ-ring, and let $T$: $K\rightarrow K$ a centralizer, then it is clear that $(T, T)$ is a double centralizer.

In the following proposition, we shall prove that the existence of additive mappings $T$, $S$: $K\rightarrow K$ fulfilling $x\,\alpha\,T(x)=S(x)\,\alpha\,x$ for any $x \in K$ $\alpha \in \Gamma$, yields that $T$- $S$ is inner derivation.

**Proposition 3.9**

Let $K$ be a Γ-ring, with identity and let $T$, S: $K\to K$ be additive mappings fulfilling $x\,\alpha\, T(x) = S(x)\,\alpha\, x$ for any $x \in$ K, α ∈ Γ. Then $T - S$ is an inner derivation.

**Proof:**

We have

$$x\;\alpha\;T(x) = S(x)\;\alpha\;x \quad \text{for all } x \in \mathrm{K}, \alpha, \beta \in \Gamma \tag{1}$$

Replacing x by $x + 1$ in (1), we get

$$x\;\alpha a + T(x) = S(x) + a\;\alpha\;x \quad \text{for all } x \in \mathrm{K}, \alpha \in \Gamma \tag{2}$$

where $T(1) = S(1) = a$.

Then from relation (2), we have

$$(T - s)(x) = a\;\alpha\;x - x\;\alpha\;\mathrm{a} = [a, x]_a \quad \text{for all } x \in \mathrm{K}, \alpha \in \Gamma$$

Hence,

$T - S$ is inner derivation.

**Definition 3.10**

Let $K$ be a $\Gamma$ -ring, and let $T, S: K \to K$, be additive mappings, then a pair $(T, S)$ is named a double Jordan centralizer, if $T$ is a left Jordan centralizer, $S$ is a right Jordan centralizer, and they satisfy a balanced requirement $x\,\alpha\, T(x) = S(x)\,\alpha\, x$, for any $x \in K$, $\alpha \in \Gamma$.

**Example 3.11**

Let $F$ be a field, and $K_2(F)$ be a Γ-ring of all 2 by 2 matrices with usual addition and multiplication of matrices, and $\Gamma = \{ \begin{bmatrix} n & 0 \\ 0 & 0 \end{bmatrix}, n \text{ is integer}\}$.

Define $T$, $S$: $K_2(F) \to K_2(F)$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}, \text{ for any } a, b, c, d \in F.$$

$$S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}, \text{ for any } a, b, c, d \in F.$$

It is clear that $T$ is a left Jordan centralizer and $S$ is a right Jordan centralizer satisfy the condition

$$x\;\alpha\;T(y) = S(x)\;\alpha\;y.$$

Therefore, $(T, S)$ is a double Jordan centralizer.

**Remark 3.12**

Every double centralizer is a double Jordan centralizer, but the opposite in general is need not to be true .

In the following example justifies this remark.

**Example 3.13**

Let $K$, $\Gamma$ and $T$ be as in the Example 2.7 and defined $S: K\to K$ by

$$S(x) = \begin{bmatrix} 0 & a & c & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ where } x = \begin{bmatrix} 0 & a & c & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is clear that $T$ and $S$ are Jordan centralizers but not centralizer, and satisfy

$$x\;\alpha\;T(x) = S\,(x)\;\alpha\;x.$$

Hence $(T, S)$ is a double Jordan centralizer but it is not double centralizer.

**Theorem 3.14**

Let $K$ be a 2-torssion free semiprime $\Gamma$ -ring, then every double Jordan centralizer is a double centralizer.

**Proof**: According to Theorem 2.9, we obtain $T$ is a left centralizer and $S$ is a right centralizer.

Let us verify that $x\ \alpha\ T(y) = S(x)\ \alpha\ y$, for any $x, y \in K$ and $\alpha \in \Gamma$.

That is, by hypothesis,

$$x\ \alpha\ T(x) = S(x)\ \alpha\ x \text{ for any } x \in K, \alpha \in \Gamma. \quad (1)$$

Now replacing $x$ by $x + y$ in (1), we get

$$x\alpha\ T(y) + y\alpha\ T(x) = S(x)\ \alpha\ y + S(y)\ \alpha\ x, \text{ for any } x, y \in K, \alpha \in \Gamma. \quad (2)$$

Setting $y = y\ \beta\ z$ in (2), we arrive at

$$x\ \alpha\ T(y\ \beta\ z) + y\ \beta\ z\ \alpha\ T(x) = S(x)\ \alpha\ y\ \beta\ z + S(y\ \beta\ z)\ \alpha\ x$$

$$(x\ \alpha T(y) - S(x)\alpha y)\ \beta\ z = y\ \beta\ (S(z)\alpha\ x - z\ \alpha T(x)), \text{ for any } x, y \in K, \alpha \in \Gamma \quad (3)$$

By replacing x with y in (3), we obtain

$$y\ \beta\ (S(z)\alpha\ y - z\ \alpha\ T(y)) = 0 \text{ for any } y, z \in K, \alpha, \beta \in \Gamma. \quad (4)$$

Putting z=z $\sigma$ x in (4), we get

$$y\ \beta\ z\ \sigma\ (S(x)\ \alpha\ y - x\ \alpha\ T(y)) = 0 \text{ for any } x, y, z \in K, \alpha, \beta, \sigma \in \Gamma. \quad (5)$$

Yields that,

$$T(y)\ \beta\ z\ \sigma(S(x)\alpha y - x\alpha T(y)) = 0, \text{ for any } x, y \in K, \alpha, \beta, \sigma \in \Gamma. \quad (6)$$

Left multiplication the relation (6) by $x$, give us

$$x\ \alpha T(y)\beta\ z\sigma(S(x)\alpha y) - x\alpha T(y)) = 0 \quad \text{for any } x, y \in K, \alpha, \beta, \sigma \in \Gamma. \quad (7)$$

Also, left multiplication the relation (5) by S (x), we get

$$S(x)\alpha y\beta\ z\sigma\big(S(x)\alpha\ y - x\ \alpha T(y)\big) = 0 \text{ for any } x, y, z \in K, \alpha, \beta, \sigma \in \Gamma. \quad (8)$$

Subtracting (7) from (8), we have

$$(S(x)\ \alpha y - x\ \alpha T(y))\beta z\sigma\ (S(x)\alpha\ y - x\alpha T(y)) = 0 \text{ for any } x, y, z \in K, \alpha, \beta, \sigma \in \Gamma. \quad (9)$$

By the semiprimness of K, we get $S(x)\alpha\ y = x\ \alpha\ T(y)$ for any $x, y \in K, \alpha \in \Gamma$.

Let us point out is case $K$ has an identity element, Theorem 3.14 can be proved for an arbitrary $\Gamma$-ring as following:

**Theorem 3.15**

Let K be a $\Gamma$ -ring with identity, then every double Jordan centralizer is a double centralizer.

For the proof of the above theorem, we need the following lemma:

**Lemma 3.16**

Let $K$ be a Γ-ring with identity element. Then, $(T, S)$ is a double Jordan centralizer if and only if $T$ and $S$ are of the from $T(x) = a\ \alpha\ x$ and $S(x) = x\ \alpha\ a$ for some fixed element $a \in K$, $\alpha \in \Gamma$.

**Proof:**

Let $(T, S)$ be a double Jordan centralizer, then

$$T(x\ \alpha\ x) = T(x)\alpha\ x \quad \text{for any } x \in K, \alpha \in \Gamma. \quad (1)$$

$$S(x\ \alpha\ x) = x\ \alpha\ S(x) \quad \text{for any } x \in K, \alpha \in \Gamma. \quad (2)$$

$$x\ \alpha\ T(x) = S(x)\ \alpha\ x \quad \text{for any } x \in K, \alpha \in \Gamma. \quad (3)$$

Replace $x$ by $x + 1$ in (1), we get

$T(x) = a\,\alpha\, x$ for any $x \in K$, $\alpha \in \Gamma$, where $a = T(1)$.
Also, replace $x$ by $x+1$ in (2), we get
$S(x) = x\,\alpha\, b$ for any $x \in K$, $\alpha \in \Gamma$, where $b = S(1)$
Now, setting $x$=1 in (3), we get a = b.
Therefore, we obtain
$T(x) = a\,\alpha\, x$ and $S(x) = x\,\alpha\, a$ for any $x \in K$, $\alpha \in \Gamma$.
To show the opposite, assume that: $T(x) = a\,\alpha\, x$ and $S(x) = x\,\alpha\, a$ for any $x \in K$, $\alpha \in \Gamma$.
Since $x\alpha T(x) = x\alpha a\,\alpha\, x = S(x)\alpha x$. Therefore, the pair ($T$, $S$) is a double Jordan centralizer.

**Proof the Theorem 3.15**

For Lemma 3.16, we get $T(x) = a\,\alpha\, x$ and $S(x) = x\,\alpha\, a$ for any $x \in K$, $\alpha \in \Gamma$. So, T is a left centralizer, $S$ is a right centralizer and
$x\,\alpha\, T(y) = x\,\alpha\, a\,\alpha\, y = S(x)\,\alpha\, y$.
Therefore, ($T$, $S$) is a double centralizer.

Now, we shall prove the following result which involves every double centralizer ($T$, $S$) of $K$ induced a derivation $d$, defined by

$$d(x) = T(x) - S(x).$$

**Remark 3. 17**

Let $K$ be a Γ-ring, then every double centralizer ($T$, $S$) of $K$ induced a derivation $d$ defined by $d(x) = T(x) - S(x)$ for any $x \in$ K.

**Proof:**

We have $d(x) = T(x) - S(x)$ for any $x \in K$.
Replace x with x α y in the above relations, we get

$$d(x\,\alpha\, y) = T(x)\,\alpha\, y - x\,\alpha\, S(y)$$
$$= \big(T(x)\alpha\, y - S(x)\alpha\, y + x\,\alpha\, T(y) - x\,\alpha S(y)\big)$$
$$= d(x)\,\alpha\, y + x\,\alpha\, d(y), \quad \text{for any } x \in K, \alpha \in \Gamma.$$

**Proposition 3.18**

Let $K$ be a Γ-ring, and let ($T_1$, $S_1$), ($T_2$, $S_2$) be double centralizers of $K$, define $d, g: K \to K$ by

$$d(x) = T_1(x) - S_2(x) \quad \text{for any } x \in K. \qquad (1)$$
$$g(x) = T_2(x) - S_1(x) \quad \text{for any } x \in K. \qquad (2)$$

Then ($d$, $g$) is a derivation pair.

**Proof :** We intend to prove the equations

$$d(x\,\alpha\, y\,\beta\, x) = d(x)\,\alpha\, y\,\beta\, x + x\,\alpha\, g(y)\,\beta\, x + x\,\alpha\, y\,\beta\, d(x)$$
$$\text{for any } x, y \in K, \alpha, \beta \in \Gamma. \qquad (3)$$
$$g(x\,\alpha\, y\,\beta\, x) = g(x)\,\alpha\, y\,\beta\, x + x\,\alpha\, d(y)\,\beta\, x + x\,\alpha\, y\,\beta\, g(x)$$
$$\text{for any } x, y \in K, \alpha, \beta \in \Gamma. \qquad (4)$$

To prove (3), putting x α y $\beta$ x for x in (1), we get

$$d(x\,\alpha\, y\,\beta\, x) = T_1(x\,\alpha\, y\,\beta\, x) - S_2(x\,\alpha\, y\,\beta\, x)$$
$$= (T_1(x) - S_2(x))\,\alpha\, y\,\beta\, x + S_2(x)\,\alpha\, y\,\beta\, x - S_2(x\,\alpha\, y\,\beta\, x)$$
$$= d(x)\alpha\, y\,\beta\, x + xT_2(y)\alpha y\,\beta\, x - x\,\alpha\, S_1(y)\beta x + x\alpha S_1(y)\,\beta x - S_2(x\,\alpha\, y\,\beta\, x)$$
$$= d(x)\,\alpha\, y\,\beta\, x + x\,\alpha\, g(y)\beta\, x + x\,\alpha\, y\,\beta\, T_1(x) - x\,\alpha\, y\,\beta\, S_1(x)$$
$$= d(x)\,\alpha\, y\,\beta\, x + x\,\alpha\, g(y)\beta\, x + x\,\alpha\, y\,\beta\, d(x)$$
$$\text{for any } x, y \in K, \alpha, \beta \in \Gamma.$$

*Analogously*, $g(x\,\alpha\, y\,\beta\, x) = g(x)\alpha\, y\,\beta\, x + x\alpha\, d(y)\,\beta\, x + x\,\alpha\, y\,\beta\, g(x)$, for any $x, y, \in$ K, $\alpha, \beta \in \Gamma$.

Thus**,** the pair ($d$, $g$) is a derivation pair.

## 4. **Conclusions**

In this work, we discussed centralizers and double centralizers in semiprime $\Gamma$ −rings with fulfilling certain identities.

**References**


**[1]** W. E. Barnes, "On the Γ-rings of Nobusawa", *Pacific J. Math*., vol. 18, pp.411-422, 1966.

**[2]** N. Nobusawa, "On a generalization of the ring theory", *Osaka J. Math*., vol. 1, pp. 81-89, 1984.

**[3]** D. Özden, M. A. Özturk and Y. B. Jun, "Permuting tri-derivations in Prime and semi-prime gamma rings", *Kyung Pook Math. J*., vol. 46, pp. 153-167, 2006.

**[4]** S. Chakraborty and A. C. Paul, "On Jordan generalized k-derivations of semiprime $\Gamma_N$-rings", *Bulletin of the Iranian Math. Soc*., vol. 36, no. 1, pp. 41-53, 2010.

**[5]** S. Kyuno, "Prime ideals in gamma rings", *Pacific J. of Math*., vol. 98, no. 2, pp. 375-379, 1982.

**[6]** S. Chakraborty and A. c. Paul, "On Jordan k-derivations of 2-torison free Prime $\Gamma_N$-rings" , *Punjab University J. of Math*, vol. 40, pp. 97-101, 2008.

**[7]** A. H. Majeed, S. K. Motashar, "Γ-Centralzing mappings of semiprime Γ-rings", Iraqi Journal of Science, vol.53, no.3, pp. 657-662, 2012.

**[8]** M.M.K.RAO, "On *Γ*-semiring with identity" General Algebra and Applications 37(2017)189-207

**[9]** Y. Ceven and M.A. Öztürk, "On Jordan centralized derivations in gamma Rings", *Hacettepe. J. Math. and statistics*, vol. 33, pp.11-14, 2004.

**[10]** A.T. Mutlak and A. H. . Majeed, "On Centralizers of 2-torsion Free Semiprime Gamma Rings", *Iraqi Journal of Science*, vol. 62, no. 7, pp. 2351–2356, Jul. 2021.

**[11]** A. H. Majeed and S.A. Hamil," γ- Orthogonal for K- Derivations and K- Reverse Derivations", *Journal of physics*,1530,2020 p.p.1-6.

**[12]** A. H. Majeed and S. Ali Hamil, "Derivations in Gamma Rings with γ-Lie and γ-Jordan Structures" 2020 *J. Phys.: Conf. Ser.* **1530** 012049.

**[13]** A. H. Majeed and S. Ali Hamil, "Derivations and reverse derivations on γ-prime and γ-semiprime gamma semirings" 2020 *J. Phys.: Conf. Ser.* **1530** 012050.

**[14]** A. K. Kadhim, H. Sulaiman , A. H. Majeed "Γ ∗ -Derivation Acting As An Endomorphism And As An Anti-Endomorphism In Semiprime Γ-Ring M With Involution", *International Journal of Pure and Applied Mathematics*, v. 102, no. 3 pp. 495-501, 2015.

**[15]** S. Chakraborty , Md. M. Rashid and A. C. Paul "Inner derivations on semiprime gamma rings", *Ganit* J. *Bangladesh Math. Soc*., vol. 39.pp. 101-110, 2019.

**[16]** A. K. Kadhim, H.Sulaiman and A. H. Majeed, "Jordan $\Gamma^*$ −centralizers and reverse Γ*-centralizers on Semiprime Γ-ring with involution", *International Math. Forum*, vol. l0, no.8, pp. 385-393, 2015.

**[17]** M.F. Hoque and A.C. Paul. "On centralizers of semiprime Gamma rings". *Int. Math Forum*, vol. 6, no.13, pp. 627-638, 2011.

**[18]** A. A. Abdulridha, A. H. Majeed, "On Γ-n- (Anti ) Generalized Strong Commutativity Preserving Maps for Semiprime **Γ**-Rings", vol. 64, no. 6, pp. 3044-3053, 2023.