DOI: 10.24996/ijs.2024.65.6.34

Some Games Via (D, DL) Compact Topological Groups

Afraa R. Sadek*[1], R. B. Esmaeel [2]

[1]Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

[2]Department of Mathematics, Ibn Al-Haitham College of Education, University of Baghdad, Baghdad, Iraq

Received: 3/5/2023 Accepted: 1/10/2023 Published: 30/6/2024

Abstract

The aim of our work is to develop a new type of games which are related to (D, WD, LD) compactness of topological groups. We used an infinite game that corresponds to our work. Also, we used an alternating game in which the response of the second player depends on the choice of the first one. Many results of winning and losing strategies have been studied, consistent with the nature of the topological groups. As well as, we presented some topological groups, which fail to have winning strategies and we give some illustrated examples. Finally, the effect of functions on the aforementioned compactness strategies was studied.

Keywords: Games, Group, Cover, D-Cover, Compact topological group.

-

بعض المباريات عبر الزمر التبولوجية المتراصة (D, DL)

عفراء راضي صادق*[1] ، رنا بهجت اسماعيل[2]

[1]قسم الرياضيات ، كلية العلوم ، جامعة بغداد، بغداد ، العراق

[2]قسم الرياضيات، كلية التربية للعلوم الصرفة (ابن الهيثم)، جامعة بغداد، بغداد، العراق

الخلاصة

الهدف من عملنا هي تطوير نوع جديد من المباريات والتي تتعلق بالتراص (D,WD,LD) للزمر التبولوجية. استخدمنا مباريات غير منتهية تتوافق مع عملنا . وكذلك استخدمنا مباريات متناوبة والتي فيها استجابة اللاعب الثاني تعتمد على اختيار اللاعب الاول. العديد من النتائج وستراتيجيات الفوز والخسارة تمت دراستهم ، بما يتوافق مع طبيعة الزمر التبولوجية. كما قدمنا بعض الزمر التبولوجية التي فشلت في امتلاك ستراتيجية رابحة وتم توضيحها من بعض خلال الامثلة. اخيرا تم دراسة تاثير الدوال على ستراتيجيات التراص سالف الذكر.

1. Introduction

Since 1900, many researchers have been working on topological groups, especially on compact topological groups [for instance see 1, 2]. Recall that, a topological group $(M,*,\tau)$ is a group and a topological space at the same time, such that the closed and inverse operations of the group are continuous, see [3], [4], and [5]. Also, a topological group $(M,*,\tau)$ is said to be compact topological group if a topological space (M,τ) is compact see [2, 6].

*Email: afraa.sadek@sc.uobaghdad.edu.iq

The investigators in [7] introduced the concept of D-cover for a topological groups as follows: the family $\{M_i \in \tau\,; (M_i,*)$ is a proper subgroup of $(M,*), \forall\, i \in I\,\}$ is called a D-cover topological groups of $(M,*,\tau)$ if $M = \cup_{i\in I} M_i$. And, by using the concept of D-cover, they defined the D-compact and DL-compact topological groups in the following way: a topological group $(M,*,\tau)$ is termed D-compact (resp., DL-compact) if for any D-cover of $(M,*,\tau)$ there is a finite sub. D-cover (resp., computable sub. D-cover).

The main objective of this research is to apply some properties of covering property in topological groups in game theory, where some types of matches are used, which have the type of endless matches, and they are between two players only, such that the profit of one is equal to loss of the other one. Further, the set of all choices made by players during the game is called a strategy. [8, 9,10]. The relationship between the winning strategies of compactness of types (*D*, *WD*, *LD)* on topological groups has been studied, and reinforced with examples of winning and non-winning strategies for the aforementioned compactness types. Several theorems of sufficient conditions for topological groups and their product which have winning strategies of the three types of compactness have been proved. In particular, the functions that preserve the winning strategies of D -compactness have been studied as an important type in our research.

We will symbolize the winning strategy by $\nearrow$, and the symbol $\searrow$ for the losing strategy and by ⤱ when the player doesn't have a winning strategy of any of the players, as well as we symbolize the first player by $\boldsymbol{p_1}$ and the second player by $\boldsymbol{p_2}$. We mean throughout this paper a topological group is just a group as a set with topology.

2. Games of D, DL compact groups

Definition 2.1: Let C be a collection of all D-cover of a topological group $(M,*,\tau)$. We define an infinite game by $\breve{G}(D,M)$ (resp., $\breve{G}(DL,M)$ as follows:

$\boldsymbol{p_1}$: Choose $M_1 \in C$.

$\boldsymbol{p_2}$: Select a nonempty finite (resp., countable).

Subset K_1 of M_1 such that $B_1 = \cup\,\{K_i : K_i \in M_1\}$. In the n-th inning $\boldsymbol{p_1}$ choose $M_n \in C$ and $\boldsymbol{p_2}$respond by selecting a non-empty finite (resp., countable) subset K_n of M_n such that $B_n = \bigcup\{\,K_i\colon K_i \in M_n\}$. $\boldsymbol{p_2}$ wins the play. M_1, B_1, M_2, B_2 ... of this game if $\{B_n\,, n \in N\}$ is D-cover (resp., DL-cover) of $(M,*,\tau)$, otherwise $\boldsymbol{p_1}$ wins.

Further, if there exists at least one finite sub-D-cover of $(M,*,\tau)$, then this game will be symbolized by $\breve{G}(WD,M)$.

Remake 2.2:

(1) If player $\boldsymbol{p_2} \nearrow \breve{G}(D,M)$, then $\boldsymbol{p_2} \nearrow \breve{G}(WD,M)$.

(2) If player $\boldsymbol{p_2} \nearrow \breve{G}(D,M)$, then $\boldsymbol{p_2} \nearrow \breve{G}(DL,M)$.

(3) If player $\boldsymbol{p_1} \nearrow \breve{G}(WD,M)$, then $\boldsymbol{p_1} \nearrow \breve{G}(D,M)$.

(4) If player $\boldsymbol{p_1} \nearrow \breve{G}(DL,M)$, then $\boldsymbol{p_1} \nearrow \breve{G}(D,M)$.

(5) If player $\boldsymbol{p_2} \nearrow \breve{G}(C,M)$, then $\boldsymbol{p_2} \nearrow \breve{G}(D,M)$.

Example 2.3: Let $M = \{0\} \cup (\cup_{n=1}^{\infty} \frac{1}{n})$, where $n \in N$ (N the set of natural numbers). Define a binary operation $*$ on M as follows:

$a * b \text{ is } \max\{a,b\}, \text{ or either one if } a = b$, and let

$\tau = \left\{\left\{0,\frac{1}{2}\right\},\left\{0,\frac{1}{2},\frac{1}{3}\right\},\left\{0,\frac{1}{2},\frac{1}{3},\frac{1}{4}\right\},\left\{1,0,\frac{1}{n},\ldots,n > 2\right\}\right\} \cup \{G,\emptyset\}.$

It is clear that τ is a topology on M. Now, let C be the collection of all D-cover and consider the game $\breve{G}(D, M)$ of $(M, *, \tau)$. Player $\boldsymbol{p_1}$ choose $A_1 \in C$ and $\boldsymbol{p_2}$ respond by find greatest n since N is well ordered set and selecting $B_1 = \{M_1, M_1'\}$, where $M_1 = \left\{0, \frac{1}{2}, \frac{1}{3}, \dots \dots . \frac{1}{n_1}\right\}$ and $M_1' = \left\{1, 0, \frac{1}{n_1+1}, \frac{1}{n_1+2}, \dots \dots .\right\}$.

In the r-th inning $\boldsymbol{p_1}$ choose $A_r \in C$ and $\boldsymbol{p_2}$ respond by find greatest n, and selecting $B_r = \{M_r, M_r'\}$, where $M_r = \left\{o, \frac{1}{2}, \dots, \frac{1}{n_r}\right\}$ and $M_r' = \left\{1, 0, \frac{1}{n_r+1}, \dots \dots\right\}$. Then $B = \{A_1, B_1, A_2, B_2, \dots \dots\}$ is the winning strategy for $\boldsymbol{p_2}$ in $\breve{G}(D, M)$. Hence, $\boldsymbol{p_2} \nearrow \breve{G}(D, M)$. In fact $(M, *)$ in this game is a semigroup.

Example 2.4: Let $M = (R^+ \cup \{0\}, *)$, where the operation $*$ is defined as follows :
$a * b = \begin{cases} \max\{a, b\} - \min\{a, b\} & a \neq b \\ 0 & a = b \end{cases}$, and let $\tau = \left\{\{0\} \cup (a, b); a, b \in R^+\right\}$.
It is clear that $(M, *, \tau)$ is a topological group and each member of τ is a proper subgroup of M. By supposing C as above, and by defining an infinite long game $\breve{G}(DL, M)$ as follows:
In the n-th inning, the player $\boldsymbol{p_1}$ choose $A_n \in C$, and by the density of rational numbers. The player $\boldsymbol{p_2}$ can be selecting a countable subset M_n of A_n such that $M_n = \left\{\{0\}, (a, b); a, b \in R^+\right\}$ which covers M. Hence, $\boldsymbol{p_2} \nearrow \breve{G}(DL, M)$.

Remark 2.5: If $M = (R^+ \cup \{0\}, *, \tau)$, as in the previous example. Players $\boldsymbol{p_1}$ and $\boldsymbol{p_2}$ are playing game $\breve{G}(D, M)$ as follows: If $\boldsymbol{p_1}$ choose $A_1 \in C$, where

$$A_1 = \{\{0\}, \cup (0,1), \{0\} \cup \{\frac{1}{2}, 2\}, \{0\} \cup \left(1\frac{1}{2}, 3\right), \dots \dots\}$$

Player $\boldsymbol{p_2}$ can not find a non-empty subset M_1 of A_1 such that M_1 is D-cover of M. Thus, there is no winning strategy for $\boldsymbol{p_2}$ in $\breve{G}(D, M)$. Hence, $\boldsymbol{p_1} \nearrow \breve{G}(D, M)$.

Remark 2.6:
Although the topological group in Example 2.4 is not a D-compact, but it is a WD-compact because of
$M = \{\{0\} \cup (0,1), \{0\} \cup (0,2), \dots \dots, \{0\} \cup (0, n), \{0\} \cup \left(n - \frac{1}{2}, \infty\right)\}$ is a finite D-cover of M.

Proposition 2.7:
(1) A topological group $(M, *, \tau)$ is a D-compact if and only if $\boldsymbol{p_2} \nearrow \breve{G}(D, M)$.
(2) If $\boldsymbol{p_2} \nearrow \breve{G}(D, M)$, then $\boldsymbol{p_2} \nearrow \breve{G}(DL, M)$.
(3) If $\boldsymbol{p_2} \nearrow \breve{G}(D, M)$, then M is a WD-compact.

Proof: Follows from the definitions and the preceding examples.

Theorem 2.8:
In any infinite not cyclic group, $\boldsymbol{p_2}$ may have a winning strategy in $\breve{G}(D, M)$.
Proof: Let $(M, *)$ be an infinite not cyclic group, and let $\tau = \{A_i \subseteq M; A_i^c$ is a finite set and $(A_i, *)$ is a subgroup $\forall i$ such that $A_{i1} \subseteq A_{i2}$ for $i_1 \leq i_2\}$.
Now, since any noncyclic group M such that $(O(M) > 4)$ has a proper subgroup [11,12], hence $\tau \neq \emptyset$ and one can show easily that τ is topology on M, thus $(M, *, \tau)$ is topological group. Consider the game $\breve{G}(D, M)$, player $\boldsymbol{p_1}$ choose $a\ D -$ cover $A_1 = \{A_{1i} \in \tau, i \in I\}$ of $(M, *, \tau)$.
player $\boldsymbol{p_2}$ responds by choosing A_{10} from A_1, hence A_{10}^c is finite say $\left\{a_{11,} a_{12}, \dots \dots \dots, a_{1n}\right\}$. Thus $M_1 = \left\{A_{10,} A_{a11,} A_{a12}, \dots \dots, A_{a1n}\right\}$.

Which is finite D-cover of $(M, *, \tau)$, since $M = A_{10} \cup A_{10}^c$ where $A_{1j}, 0 \leq j \leq n$ are members of τ containing $a_{11}, a_{12}, \ldots. a_{1n}$, respectively.
In the second inning player $\boldsymbol{p_1}$ choose a D-cover $A_2 = \{A_{2i}; A_{2i} \in \tau$, i∈ $I\}$ of $(M,*, \tau)$. Player $\boldsymbol{p_2}$ responds by fixed $A_{20} \in \{A_{2i}\}_{i \in I}$. Thus $A_{20}^c = \{a_{21}, a_{22}, \ldots. a_{2n}\}$ and selecting $M_2 = \{A_{20}, A_{a21}, \ldots. A_{a2n}\}$ which is finite D-cover of $(M,*, \tau)$.
In the r-th inning similarly player $\boldsymbol{p_1}$ choose a D-cover $A_r = \{A_{ri}, A_{ri} \in \tau$, $i \in I\}$ of $(M,*, \tau)$. Player $\boldsymbol{p_2}$ responds by fixed $A_{r0} \in \{A_{ri}\}_{i \in I}$, thus $A_{r0}^c = \{a_{r1}, a_{r2}, \ldots \ldots, a_{rn}\}$ and selecting $M_r = \{A_{r0}, A_{ar1}, \ldots. A_{arn}\}$ which is finite D-cover of $(M,*, \tau)$.
Thus player $\boldsymbol{p_2}$ has a winning strategy $B = \{A_1, M_1, A_2, M_2, \ldots , A_r, M_r, \ldots\}$ and can be win the game $\breve{G}(D, M)$.

Corollary 2.9:
In any infinite not cyclic group, we have $\boldsymbol{p_2} \nearrow \breve{G}(WD, M)$.

Remark 2.10:
In any cyclic group, then $\boldsymbol{p_2} \not\nearrow \breve{G}(D, M)$.

Proof: Suppose $(M, *, \tau)$ be a topological group where $(M, *)$ is infinite cyclic group. Now, since any infinite cyclic group is isomorphic to additive group of integers (Z, +) [see 13] , so let C be the collection of all open cover of M and consider the game $\breve{G}(D, M)$ on $M = (Z, +, \tau)$. Player $\boldsymbol{p_1}$ and player $\boldsymbol{p_2}$ are playing in this game. Then in the first inning player $\boldsymbol{p_1}$ try to choose a D-cover $A = \{< m >; m \in Z\}$ because every subgroup of a cyclic group is cyclic. But the generator $\{\pm 1\}$ of (Z, +) can not belong to any proper subgroup [14]. Thus, there is no D-cover of $(Z, +, \tau)$. Hence $\boldsymbol{p_2} \not\nearrow \breve{G}(D, M)$. We will obtain the same result if $(M, *)$ was a finite cyclic group.

Theorem 2.11:
A topological group $(M,*, \tau)$ is a D-compact if and only if $\boldsymbol{p_2} \nearrow \breve{G}(D, M)$.

Proof: Suppose $(M,*, \tau)$ is D-compact, for any choose D-cover A of player $\boldsymbol{p_1}$, then player $\boldsymbol{p_2}$ will be find a finite sub- D-cover of A. Hence $\boldsymbol{p_2} \nearrow \breve{G}(D, M)$. For sufficiency if player $\boldsymbol{p_2} \nearrow \breve{G}(D, M)$, this means if A any arbitrary D-cover of $(M,*, \tau)$ which is choose by $\boldsymbol{p_1}$, then player $\boldsymbol{p_2}$ can be choose a finite sub-D-cover of A, thus $(M,*, \tau)$ is a D-compact.

Corollary 2.12:
A topological group $(M,*, \tau)$ is a D-compact if and only if player $\boldsymbol{p_1} \not\nearrow \breve{G}(D, M)$.

Proof: Suppose $(M,*, \tau)$ is a D-compact.
If $\boldsymbol{p_1} \nearrow \breve{G}(D, M)$, there is a D-cover of $(M,*, \tau)$ which has no finite sub-D-cover, but this a contradiction with our assumption.
Now, suppose player $\boldsymbol{p_1} \not\nearrow \breve{G}(D, M)$ and $(M,*, \tau)$ is not a D-compact, then there exist at least one D-cover say A' of $(M,*, \tau)$ which has no finite sub-D-cover. Thus player $\boldsymbol{p_1}$ can choose A' and winning the game which is impossible since player $\boldsymbol{p_1} \not\nearrow \breve{G}(D, M)$.

3. Games via product and mapping between topological groups

Let $(M_{\gamma,*_\gamma,\tau_\gamma})$ be a topological group, for each $\gamma \in \Lambda$. By $(\pi_{\Upsilon \in \Lambda} M_\gamma, \circledast, \pi_{\Upsilon \in \Lambda} \tau_\gamma)$ we mean the product of $\{M_\gamma\}_{\Upsilon \in \Lambda}$ equipped with usual product topology [see 15,16] and the multiplication is given by
$(x \circledast y) = x_\gamma \circledast_\gamma y_\gamma$ for each $x_\gamma, y_\gamma \in M_\gamma$.

For product topological groups we have the following theorem

Theorem 3.1:
If $(M,*,\tau)$ and $(N,\overline{*},\overline{\tau})$ are two topological groups and if $\boldsymbol{p_2} \nearrow \breve{G}(D,N)$, then $\boldsymbol{p_2} \nearrow \breve{G}(D,M\times N)$.

Proof: Consider we have the game $\breve{G}(D,M\times N)$. Player $\boldsymbol{p_1}$ choose a D-cover $\beta_1 = \{(M\times N_{1i},\circledast)\ ;\ N_{1i} \in \overline{\tau}$ and $(N_{1i},\overline{*})$ is a subgroup $\forall i \in I\}$.
But $M\times N = \cup_{i\in I}(M\times N_{1i}) = M\times(\cup_{i\in I} N_{1i})$ hence
$N = \cup_{i\in J_1} N_{1i}$ and since $\boldsymbol{p_2} \nearrow \breve{G}(D,N)$ so, there is a finite subset $J_1 \subseteq I$ such that
$N = \cup_{i\in J_1} N_{1i}$, hence $M\times N = M\times(\cup_{i\in J_1} N_{1i}) = \cup_{i\in J_1}(M\times N_{1i})$.
In the second inning player $\boldsymbol{p_1}$ choose a D-cover
$\beta_2 = \{(M\times N_{2i},,\circledast)\ ;\ N_{2i} \in \overline{\tau}, (N_{2i},\overline{*})$ is a group $\forall i \in I\}$.
But $M\times N = \cup_{i\in I}(M\times N_{2i}) = M\times(\cup_{i\in I} N_{2i})$, hence
$N = \cup_{i\in I} N_{2i}$ and since $\boldsymbol{p_2} \nearrow \breve{G}(D,N)$ so, there is a finite subset $J_2 \subseteq I$ such that
$N = \cup_{i\in J_2} N_{2i}$, hence $\times N = M\times(\cup_{i\in J_2} N_{2i}) = \cup_{i\in J_2}(M\times N_{2i})$.
In the r-th inning player $\boldsymbol{p_1}$ choose a D-cover
$\beta_r = \{(M\times N_{ri},\circledast)\ ;\ N_{ri} \in \overline{\tau}$ and $(N_{ri},\overline{*})$ is a subgroup $\forall i \in I\}$. Thus similarly as the first and second inning player $\boldsymbol{p_2}$ can select a finite subset $J_r \subseteq I$ such that
$N = \cup_{i\in J_r} N_{ri}$, implies $M\times N = \cup_{i\in J_r}(M\times N_{ri})$.
Thus player $\boldsymbol{p_2}$ has a winning strategy $\beta = \{\beta_1, J_1, \beta_2, J_2, \ldots\ldots.\}$ which complete the proof.
Recall that an isomorphism of a topological group is a group isomorphism that is also a homeomorphism of the underlying topological space [17].

Theorem 3.2:
Let $f:(M,*,\tau) \to (N,\overline{*},\overline{\tau})$ be an isomorphism between two topological groups $(M,*,\tau)$ and $(N,\overline{*},\overline{\tau})$, and let E and L be any arbitrary subgroup of M and N respectively, then
1) If $\boldsymbol{p_2} \nearrow \breve{G}(D,E)$, then $\boldsymbol{p_2} \nearrow \breve{G}(D,f(E))$.
2) If $\boldsymbol{p_2} \nearrow \breve{G}(D,L)$, then $\boldsymbol{p_2} \nearrow \breve{G}(D,f^{-1}(L))$.

Proof: (1) In the first inning player $\boldsymbol{p_1}$ in $\breve{G}(D,f(E))$ choose a D-cover $M_1 = \{N_{1i}\}_{i\in I}$ of $f(E)$ in $(N,\overline{*},\overline{\tau})$ that is $f(E) = \cup_{i\in I} N_{1i}$ implies
$E = f^{-1}(\cup_{i\in I} N_{1i}) = \cup_{i\in I} f^{-1}(N_{1i})$, thus $\{f^{-1}(N_{1i})\}_{i\in I}$ is a D-cover of E since f is continuous and $N_{1i} \in \overline{\tau}$ for each $i \in I$, [18].
But $\boldsymbol{p_2} \nearrow \breve{G}(D,E)$, so there is a finite subset $J_1 \subseteq I$ such that$E = f^{-1}(\cup_{i\in J_1} N_{1i})$, hence $f(E) = f(f^{-1}(\cup_{i\in J_1} N_{1i})) = \cup_{i\in J_1} N_{1i}$ [19].
For player $\boldsymbol{p_2}$ in $\breve{G}(D,f(E))$ choose $N_1 = \{N_{1i}\}_{i\in J_1}$.
In the second inning player $\boldsymbol{p_1}$ in $\breve{G}(D,f(E))$ choose a D-cover $M_2 = \{N_{2i}\}_{i\in J_1}$ of $f(E)$ in $(N,\overline{*},\overline{\tau})$ that is $f(E) = \cup_{i\in I} N_{2i}$ implies $E = f^{-1}(\cup_{i\in I} N_{2i}) = \cup_{i\in I} f^{-1}(N_{2i})$. Since f is continuous and $N_{2i} \in \overline{\tau}$ for each $i \in I$, then $f^{-1}(N_{2i})$ construct a D-cover of E.
But $\boldsymbol{p_2} \nearrow \breve{G}(D,S)$, so there is a finite subset $J_2 \subseteq I$ such that $\{f^{-1}(N_{2i})\}_{i\in J_2}$is finite sub. D-cover of E, hence $f(E) = f(f^{-1}(\cup_{i\in J_2} N_{2i}))= \cup_{i\in J_2} N_{2i}$.
For player $\boldsymbol{p_2}$ in $\breve{G}(D,f(E))$ select $N_2 = \{N_{2i}\}_{i\in j_2}$.
In the r-th inning of the game $\breve{G}(D,f(E))$ player $\boldsymbol{p_1}$choose a D-cover $M_r = \{N_{ri}\}_{i\in I}$ of $f(E)$ in $(N,\overline{*},\overline{\tau})$ means $f(E) = \{N_{ri}\}$ implies $E = f^{-1}(\cup_{i\in I} N_{ri}) = \cup_{i\in I} f^{-1}(N_{ri})$, again by the continuity of f and $N_{ri} \in \overline{\tau}, \forall i \in I$, we have $\{f^{-1}N_{ri}\}_{i\in I}$ is a D-cover of E.

But $\boldsymbol{p_2} \nearrow \breve{G}(D, f(E))$, thus player $\boldsymbol{p_2}$ can choose a finite set of I say J_r such that $\{f^{-1}(N_{ri})\}_{i\in J_r}$ is a D-cover of E.
Now since $E = f^{-1}(\cup_{i\in J_r} N_{ri})$ implies $f(E) = f(f^{-1}(\cup_{i\in J_r} N_{ri})) = \cup_{i\in J_r} N_{ri}$, so player $\boldsymbol{p_2}$ can choose $N_r = \{N_{ri}\}_{i\in J_r}$, then $B = \{M_1, N_1, M_2, N_2, \dots, M_r, N_r, \dots\}$ is the winning strategy for $\breve{G}(D, f(E))$. Hence $\boldsymbol{p_2} \nearrow \breve{G}(D, f(E))$.
(2) Suppose $\boldsymbol{p_2} \nearrow \breve{G}(D, L)$, where L as in the assumption and consider the game
$\breve{G}(D, f^{-1}(L))$.
In the first inning player $\boldsymbol{p_1}$ choose a D-cover $R_1 = \{M_{1i}\}_{i\in I}$ of $f^{-1}(L)$ in $(M, *, \tau)$ that is $f^{-1}(L) = \cup_{i\in I} \{M_{1i}; M_{1i} \in \tau, \forall i \in I\}$ implies $L = f(\cup_{i\in I} M_{1i}) = \cup_{i\in I} f(M_{1i})$.
Thus $\{f(M_{1i})\}_{i\in I}$ is a D-cover of L since f^{-1} is continuous (f homeomorphism) and $f(M_{1i}) \in \bar{\tau}$ for each $i \in I$. But $\boldsymbol{p_2} \nearrow \breve{G}(D, L)$, so There is a finite subset $J_1 \subseteq I$ such that $L = \cup_{i\in J_1} f(M_{1i})$, hence $f^{-1}(L) = \cup_{i\in J_1} (M_{1i})$.
For player $\boldsymbol{p_2}$ in $\breve{G}(D, f^{-1}(L))$ choose $E_1 = \{M_{i\in I}\}_{i\in J_1}$.
Similarly, in the n-th inning player $\boldsymbol{p_1}$ in $\breve{G}(D, f^{-1}(L))$ choose a D-cover $R_n = \{M_{ni}\}_{i\in I}$ of $f^{-1}(L)$ in $(M, *, \tau)$, means $f^{-1}(L) = \cup_{i\in I} M_{ni}$, hence $L = f(\cup_{i\in I} M_{ni}) = \cup_{i\in I} f(M_{ni})$ and since f is isomorphism, implies $f(M_{ni}) \in \bar{\tau}$ for each $i\epsilon I$.
Now, $\boldsymbol{p_2} \nearrow \breve{G}(D, L)$, so player $\boldsymbol{p_2}$ has a finite subset $J_n \subseteq I$ such that $L = (\cup_{i\in J_n} M_{ni})$, hence $f^{-1}(L) = \cup_{i\in J_n} M_{ni}$ [20], so player $\boldsymbol{p_2}$ can choose.
$E_n = \{M_{ni}\}_{i\in J_n}$. Thus, $\boldsymbol{p_2} \nearrow \breve{G}(D, f^{-1}(L))$ by selecting the winning strategy
$\beta = \{R_1, E_1, R_2, E_2, \dots \dots . R_n, E_n, \dots \dots\}$.
Note that the D-compactness is topology property as we will show in the following theorem.

Theorem 3.3:
Let $f: (M, *, \tau) \to (N, \bar{*}, \bar{\tau})$ be an isomorphism map between $(M, *, \tau)$ and $(N, \bar{*}, \bar{\tau})$, then the following are equivalent:
1) $\boldsymbol{p_2} \nearrow \breve{G}(D, M)$.
2) $\boldsymbol{p_2} \nearrow \breve{G}(D, N)$.

Proof: $(1 \Longrightarrow 2)$ Suppose $\boldsymbol{p_2} \nearrow \breve{G}(D, M)$.
In the first inning the player $\boldsymbol{p_1}$ choose a D-cover $L_1 = \{N_{1i}\}_{i\in I}$ of the topological group $(N, \bar{*}, \bar{\tau})$, that's mean each of $\{N_{1i}\}_{i\in I}$ is a subgroup and belongs to $\bar{\tau}$ for each $i \in I$. Further, $N = \cup_{i\in I} N_{1i}$ and since f is an isomorphism, we get $f^{-1}(\cup_{i\in I} N_{1i}) = \cup_{i\in I} f^{-1}(N_{1i})$. So the family $\{f^{-1}(N_{1i})\}_{i\in I}$ constructs a D-cover of M.
But $\boldsymbol{p_2} \nearrow \breve{G}(D, M)$. So, there is a finite set $J_1 \subseteq I$ such that $M = \cup_{i\in J_1} f^{-1}(N_{1i})$ implies $N = f(M) = f(f^{-1} \cup_{i\in j_1} N_{1i})) = \cup_{i\in J_1} N_{1i}$.
For player $\boldsymbol{p_2}$ in $(N, \bar{*}, \bar{\tau})$ choose $O_1 = \{N_{1i}\}_{i\in J_1}$.
In the second inning player $\boldsymbol{p_1}$in $\breve{G}(D, N)$ choose another D-cover of $(N, \bar{*}, \bar{\tau})$
$L_2 = \{N_{2i}\}_{i\in I}$, which mean $N = \cup_{i\in I} N_{2i}$ and since f is an isomorphism we have
$M = f^{-1}(N) = \cup_{i\in I} (f^{-1}(N_{2i}))$ and $f^{-1}(N_{2i}) \in \tau$ for each i, hence $f^{-1}(N_{2i})$ make up a D-cover of $(M, *, \tau)$. But $\boldsymbol{p_2} \nearrow \breve{G}(D, M)$, so there is a finite sub-D-cover of $f^{-1}(N_{2i})$ say $\{f^{-1}(N_{2i})\}_{i\in J_2}, J_2$ finite subset of I.
Hence, $M = \cup_{i\in J_2} f^{-1}(N_{2i})$ implies $N = f(M) = f(f^{-1} \cup_{i\in J_2} N_{2i})) = \cup_{i\in J_2} N_{2i}$.
Thus $\{N_{2i}\}_{i\in J_2}$ is a finite sub D-cover of N. For player $\boldsymbol{p_2}$ in $(N, \bar{*}, \bar{\tau})$ choose $O_2 = \{N_{2i}\}_{i\in J_2}$.
In the r-th inning player $\boldsymbol{p_2}$ in $\breve{G}(D, N)$ choose a D-cover of N say $L_r = \{N_{ri}\}_{i\in I}$.
Hence, $N = \cup_{i\in I} N_{ri}$ and the some causes in the previous inning we have $M = f^{-1}(N) = \cup_{i\in I} f^{-1}(N_{ri})$, but f is an isomorphism, so $f^{-1}(N_{ri}) \in \tau$, for each i.

Thus $\{f^{-1}(N_{ri})\}_{i\in I}$ is a D-cover of $(M, *, \tau)$. But $\boldsymbol{p_2} \nearrow \breve{G}(D, M)$, then there exists a finite subset $J_r \subseteq I$ such that $M = \cup_{i\in J_r} f^{-1}(N_{ri})$, hence $f(M) = N = f\left(\cup_{i\in J_r} f^{-1}(N_{ri})\right) = f\left(f^{-1}\left(\cup_{i\in J_r} N_{ri}\right)\right) = \cup_{i\in J_r} N_{ri}$, so $\{N_{ri}\}_{i\in J_r}$is a finite D-cover of $(N, \bar{*}, \bar{\tau})$. Thus player $\boldsymbol{p_2}$can select $O_r\{N_{ri}\}_{i\in J_r}$ and wins in this inning, then $B = \{L_1, O_1, L_2, O_2, \ldots\ldots L_r, O_r, \ldots.\}$
is the winning strategy for player $\boldsymbol{p_2}$ in M (D, N) implies $\boldsymbol{p_2} \nearrow \breve{G}(D, N)$.

proof: (2⟹1): Since the image of a subgroup is a subgroup under isomorphism, thus the proof follows immediately.

4. Conclusions

In this research, we adapt this theory with the (*D, WD, LD*) compactness of topological groups. By using the principle of profit and loss for alternative games, we have demonstrated many important results. To name a few: The topological groups are compact if and only if the second player wins the game and that any group which is finite and not cyclic has a winning strategy for compactness. We also studied this compactness with the multiplication of many topological groups and proved that with a winning strategy for a topological group, there is a winning strategy for its product with arbitrary topological group. We also presented many results related to compactness of *D* and *LD*.

References

[1] L.S. Pontryagin, “Topological Groups,” *Gordon and Breach Science Publishers, New York*, 1986

[2] K. Chandraskharan, “A Course on Topological Groups,” *Hindustan Book Agency, India*, 1996.

[3] S.A. Morris, “Topological Groups, Advances, Survey and Open Questions,” *La Trobe University and Federation University, Australia*, 2010.

[4] K.H. Hoffman and S.A. Morris, “The Structure of Compact Groups,” *Walter de Gruyter, Berlin, Boston*, 2013.

[5] T. Hussein, “Introduction to Topological Groups,” *Manufactured By LSC Communication. United States*, 2018.

[6] A. Adhikari and M.R. Adhikari, “Basic Topology2, Topological groups, Topology of Manifolds and Lie Groups,” *Springer, pte, ltd. Singapore*, 2022.

[7] D.G. AL-Kafajy, and A.R Sadek, “On D-Compact Topological Group.” *Iraqi Journal of Science*, vol. 54, no.3, pp. 842-846, 2013.

[8] R.E. Radwan, E. El Seidy and R.B. Esmaeel, “Infinite Games via Covering Properties in Ideal Topological Spaces,” *International Journal of Pure and Applied Mathematics*, vol.106, no. 1, 2016.

[9] R.J. Aumann, “Survey of Repeated Game,” *Essays in Game Theory and Mathematical Economics in Honor of Oscar Morgenstern*, pp.259-271, 1981.

[10] J.S. Banks, and R.K. Sundaram. “Repeated Games, Finite Automata and Complexity.” *Games and Economic Behavior*, vol. 2, pp. 97-117, 1990.

[11] D.M Burton, “Abstract and linear Algebra,” *Addision –Wesely company , Inc*,1972.

[12] D. Bogopolski, “Introduction to Group Theory,” *European Mathematical society, Germany*, 2008.

[13] A.R. Sadek, “P,PL- Compact Topological Ring,” *Iraqi Journal of Science*, vol. 57, no. 4B, pp. 2754-2759, 2016.

[14] T.H. Majeed, “Action of Topological Groupoid on Topological Space,” *International Journal on Nonlinear Analysis and Application*, vol.13, no. 1, pp. 85-89, 2022.

[15] A.R. Sadek and J.H. Hussein, “Regularity via Semi-Generalized Open Set,” *Journal of Xi, a University of Architectur & Technology*, vol. 2, no. 4, pp.4631-4636, 2020.

[16] J.H. Bayati, “On Topology Generated by Subbase on Function Space,” *AIP Conference proccedings*, 2394, 070039, 2022.

[17] W.W. Comfort, “Topological Groups,” *Hand Book of Set-Theoretic Topology. North-Holland*, pp. 1143-1263, 1948.

[18] H.J. Ali and T.I. Mahmood, “New Generalization of Soft Lc-Space.” *Iraqi Journal of Science*, vol. 63, no. 11, pp. 4890-4900, 2022.

[19] A.A. Ali and A.R. Sadek, “On Regular δ- Semi-Open Spaces.” *Journal of interdisciplinary Mathematics*, vol. 24, no.4, pp. 953-960, 2022.

[20] M.A. Hussain and Y.Y. Yousif, “Fiberewise Fuzzy Separation Axioms,” *Journal of physics Conference Series*, vol. 2322, no.1, 2022.