DOI: 10.24996/ijs.2024.65.12.27

ISSN: 0067-2904

$L_p(\mathcal{D})$ Approximation by Modified Complex Szasz- Mirakjan Operator

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Received: 1/5/2023 Accepted: 2/12/2023 Published: 30/12/2024

Abstract

Many operators are introduced for the approximation of real functions, but little for the approximation of complex functions. These studies are for analytic functions. In this article, we define a new type of Szasz-Mirakjan operator. Then we estimate the degree of approximation using this modified complex Szasz-Mirakjan operators to integrable functions on compact disks along with a quantitative estimation. Using quantitative methods, we also obtain an upper estimate in the simultaneous approximation by $\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})$ and exact degrees of approximation estimation for these operators' Stancu-type generalization.

Keywords: Exact order of approximation, rate of convergence, complex Szasz-Stancu type operator, Voronovskaja's theorem.

التقريب $L_p(\mathcal{D})$ بواسطة مؤثر سزاسز ميراكجان العقدي المعدل

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الخلاصة:

قُدم العديد من المؤثرات لتقريب الدوال الحقيقية، لكن القليل منها كان لتقريب الدوال المعقدة. هذه الدراسات للدوال التحليلية. في هذا البحث، نحنُ عرفنا نوعاً جديداً من مؤثر سزاسز ميراكجان. نحن نقدر درجة التقريب باستخدام سزاسز ميراكجان العقدي المعدل للدوال القابلة للتكامل على القرص المرصوص. باستخدام الطرق الكمية حصلنا على تقدير اعلى في التقريب المتزامن بواسطة $\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})$ والدرجات التامة لتقدير التقريب لهذا المؤثر، لتعميم نوع− ستانكو.

1. Introduction

Deeba [1], Bohman [2], Gergen et al. [3] and wood [4] investigated the several approximation properties of the complex Szasz- Mirakjan (SM) operators specified by

$$\Psi_m(\mathfrak{f})(z) = e^{-mz}\sum_{\zeta=0}^{\infty}\frac{(mz)^{\zeta}}{\zeta!}\mathfrak{f}\left(\frac{\ell}{m}\right),\ z\in\mathbb{C}.$$

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In these complex domains. Notice that in the mentioned works, the convergence results were obtained without the use of any quantitative estimates to the complex (SM) operators. Gal [5, 6, 7] recently obtained, for complex (SM) operators attached to analytic functions with different conditions on compact disks.

The order of simultaneous approximation and Voronovskaja- type results with quantitative estimate were studied in [5, 6, 7].

The (SM) operators are modified in the real case, and the approximation properties of these modified operators were studied in [8 -13].

Define $L_p(\mathcal{D}) = \left\{ f : \mathcal{D} \longrightarrow \mathbb{C} : \|f\|_{L_p(\mathcal{D})} = \left(\int_{\mathcal{D}} |f|^p \right)^{\frac{1}{p}} < \infty \right\}$ where $0 < p < 1$ and $\mathcal{D} = \{z \in \mathcal{D}_r : |z| \leq r, 1 < r < \mathcal{R}\}$.

Below we define a modification of the complex Szasz-Mirakjan operators in [9]:

$$\mathcal{M}_m(\mathfrak{f})(z) = e^{z/x_m} \sum_{\zeta=0}^{\infty} \mathfrak{f}(\zeta x_m) \frac{z^\zeta}{\zeta!\, x_m^\zeta}, \qquad z \in \mathbb{C}, \tag{1}$$

where x_m is convergent sequence.

Everywhere in the paper, $\mathfrak{f} : \mathcal{D} \longrightarrow \mathbb{C}$ which integrable in compact disk $\{z \in \mathcal{D}_r : |z| \leq r, 1 < r < \mathcal{R}\}$.

It is obvious that $\mathcal{M}_m(\mathfrak{f})(z)$ is well defined for all $z \in \mathbb{C}$.

Let us define the stancu type generalization of the operator (1) is

$$\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z) = e^{z/x_m} \sum_{\zeta=0}^{\infty} \mathfrak{f}\left(\frac{(\zeta+\delta) m x_m}{m+\gamma} \right) \frac{z^\zeta}{\zeta!\, x_m^\zeta}, \qquad z \in \mathbb{C}, \tag{2}$$

where $0 \leq \delta \leq \gamma, \delta, \gamma$ are real numbers independent of m and x_m is convergent sequence in definition $\mathcal{M}_m(\mathfrak{f})(z)$.

The goal of this article is to prove approximation results using operators given by (2) for functions in $L_p(\mathcal{D})$.

We first provide the degree of approximation and the voronovskaja type theorems with quantitative estimates for the operators $\mathcal{M}_m(\mathfrak{f})$ and $\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})$ to achieve this goal.

These results allow us to identify the precise degree of approximation used by the operators $\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})$ in (2).

2. Approximation by the operators $\mathcal{M}_m(\mathfrak{f})$

We require the subsequent auxiliary lemmas in order to establish the following results.

Lemma 1: For all $m, \ell \in \mathbb{N} \cup \{0\}, 0 \leq \delta \leq \gamma, z \in \mathbb{C}$, we have

$$\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z) = \sum_{\zeta=0}^{\infty} \left(\frac{1}{x_m(m+\gamma)} \right)^\zeta \left[\frac{\delta m x_m}{m+\gamma}, \frac{(\delta+1) m x_m}{m+\gamma}, \dots, \frac{(\delta+\zeta) m x_m}{m+\gamma}; \mathfrak{f} \right] z^\zeta \tag{3}$$

where the divided difference of the function $\mathfrak{f}$ on the knots $y_0, y_1, \dots, y_n$ is denoted by $[y_0, y_1, \dots, y_n; \mathfrak{f}]$.

Proof: By performing similar algebraic calculations in [14] yields the formula (3). It should be noted that Lupas proved such a formula in [12] for classical Szasz operators in the real case, but the formula is also valid in a complex setting [see also 7].

Lemma 2: Let $w_\ell(z) = z^\ell, \mathcal{M}_m^{(\delta,\gamma)}(w_\ell)(z) := \mathcal{H}_{m,\ell}^{(\delta,\gamma)}(z), \ell, m \in \mathbb{N}$ and x_m be convergent sequence. Then the recurrence relation

$$\mathcal{H}_{m,\ell+1}^{(\delta,\gamma)}(z) = \frac{mx_m z}{m+\gamma}\left(\mathcal{H}_{m,\ell}^{(\delta,\gamma)}(z)\right)' + \left(\frac{m(\delta x_m + z)}{m+\gamma}\right)\mathcal{H}_{m,\ell}^{(\delta,\gamma)}(z) \qquad (4)$$

is satisfied.

Proof: All $|z| \le r$ when considering equality (3) and using the divided difference mean value theorem for $f(z) = w_\ell(z) = z^\ell$ we have

$$\left\|\mathcal{M}_m^{(\delta,\gamma)}(w_\ell)(z)\right\|_{L_p(\mathcal{D})} \left\|\mathcal{H}_{m,\ell}^{(\delta,\gamma)}(z)\right\|_{L_p(\mathcal{D})} \le$$

$$\sum_{\zeta=0}^{\infty}\left(\frac{1}{x_m(m+\gamma)}\right)^\zeta \left\|\left[\frac{\delta m x_m}{m+\gamma}, \frac{(\delta+1)mx_m}{m+\gamma}, \dots, \frac{(\delta+\zeta)mx_m}{m+\gamma}; f\right]\right\| \|z^\zeta\|_{L_p(\mathcal{D})}$$

$$\sum_{\zeta=0}^{\infty}\left(\frac{1}{x_m(m+\gamma)}\right)^\zeta \left|\frac{\ell(\ell-1)\dots(\ell-\zeta+1)}{\zeta!}\right| \|z^{\ell-\zeta}z^\zeta\|_{L_p(\mathcal{D})}$$

$$\le \|z^\ell\|_{L_p(\mathcal{D})}\sum_{\zeta=0}^{\infty}\left(\frac{1}{x_m(m+\gamma)}\right)^\zeta \binom{\ell}{\zeta} \le \|z^\ell\|_{L_p(\mathcal{D})}\sum_{\zeta=0}^{\infty}\frac{1}{(mx_m)^\zeta}\binom{\ell}{\zeta}$$

$$\le 2^\ell\|z^\ell\|_{L_p(\mathcal{D})} = 2^\ell\left(\int_{\mathcal{D}} |z^\ell|^p dz\right)^{\frac{1}{p}} \le 2^\ell\left(\int_{\mathcal{D}} r^{\ell p} dz\right)^{\frac{1}{p}} = (2r)^\ell(\pi r^2)^{\frac{1}{p}}. \qquad (5)$$

Using the formula to differentiate with respect to $z \neq 0$, we obtain

$$\frac{d}{dz}\left[e^{-z/x_m}\sum_{\zeta=0}^{\infty}\left((\zeta+\delta)mx_m\right)^\ell \frac{z^\zeta}{\zeta!\,x_m^\zeta}\right]$$

$$= \sum_{\zeta=0}^{\infty}\left((\zeta+\delta)mx_m\right)^\ell\left[\frac{-e^{-z/x_m}z^\zeta}{\zeta!\,(x_m)^{\zeta+1}} + e^{-z/x_m}\zeta\frac{z^{\zeta-1}}{\zeta!\,(x_m)^\zeta}\right],$$

following which the formula is divided by $(m+\gamma)^{\ell+1}$ and using simple and direct math, we obtain

$$\frac{mx_m z}{m+\gamma}\left(\mathcal{H}_{m,\ell}^{(\delta,\gamma)}(z)\right)' = \mathcal{H}_{m,\ell+1}^{(\delta,\gamma)}(z) - \left(\frac{m(\delta x_m+z)}{m+\gamma}\right)\mathcal{H}_{m,\ell}^{(\delta,\gamma)}(z).$$

Thus, we arrive at the recurrence formula proposed by (4).

We provide the recurrence formula for the operator in the special case $\delta = 0 = \gamma$ in degree to show the key outcomes from this section (1).

Let $\mathcal{M}_m(w_\ell)(z) = \mathcal{H}_{m,\ell}(z)$ with $(w_\ell)(z) = z^\ell$ for this purpose. It is clear that $\mathcal{H}_{m,\ell}(z)$ is a polynomial with degree ℓ, where $\ell = 0, 1, 2, \dots$, and $\mathcal{H}_{m,o}(z) = 1$ and $\mathcal{H}_{m,1}(z) = z$, respectively, for all $z \in \mathbb{C}$. Using (4) and $\delta = 0 = \gamma$, we obtain

$$\mathcal{H}_{m,\ell+1}(z) = x_m z\mathcal{H}_{m,\ell}'(z) + z\mathcal{H}_{m,\ell}(z)$$

$\ell = 0, 1, 2, \dots, m \in \mathbb{N}$ for all $z \in \mathbb{C}$. This allows us to create the following formula

$\mathcal{H}_{m,\ell}(z) - z^{\ell} = x_m z[\mathcal{H}_{m,\ell-1}(z) - z^{\ell-1}]' + z[\mathcal{H}_{m,\ell-1}(z) - z^{\ell-1}] + x_m(\ell - 1)z^{\ell-1}$, (6)
for all z in the domain $\mathcal{D}$, $\ell, m \in \mathbb{N}$.

Lemma 3 [7]: In the case of classical complex Szasz--Mirakjan operator, we can write

$$\mathcal{M}_m(\mathfrak{f})(z) = \sum_{\ell=0}^{\infty} |a_\ell| \mathcal{H}_{m,\ell}(z).$$

Lemma 4 [15] (Bernstein's inequality): For any trigonometric polynomial $\mathcal{H}_{m,\ell}(z)$ of order $\leq \ell$, for every P $(1 \leq P \leq \infty)$, we have

$$\left\| [\mathcal{H}_{m,\ell}(z)]' \right\|_P \leq c\ell \left\| \mathcal{H}_{m,\ell}(z) \right\|_P, \tag{7}$$

where c is a positive constant.

Theorem 1*:* Let $\mathcal{D}_\mathcal{R} = \{z \in \mathbb{C} : |z| < \mathcal{R}, 2 < \mathcal{R} < +\infty\}$ and $\mathfrak{f} \in \mathrm{L_p}(\mathcal{D})$, $\mathfrak{f} : [\mathcal{R}, +\infty) \cup \mathcal{D}_\mathcal{R} \longrightarrow \mathbb{C}$, that is $\mathfrak{f}(z) = \sum_{\ell=0}^{\infty} a_\ell z^\ell$ for any $z \in \mathcal{D}_\mathcal{R}$. If $1 < r < \mathcal{R}/2$, then for all $|z| \leq r$, $\mathcal{D} = \{z \in \mathcal{D}_r : |z| \leq r, 1 < r < \mathcal{R}/2\}$ and $m \in \mathbb{N}$

$$\|\mathcal{M}_m(\mathfrak{f}) - \mathfrak{f}\|_{\mathrm{L_p}(\mathcal{D})} \leq x_m \mathcal{B}(\mathfrak{f}),$$

where $\mathcal{B}(\mathfrak{f}) = r\sum_{\ell=2}^{\infty}(c+1)(2\pi r^2)^{\frac{1}{p}}|a_\ell|\,\ell(2r)^{\ell-1} < \infty$.

Proof*:* By using definition $\mathrm{L_p}(\mathcal{D})$, we get

$\|\mathcal{M}_m(\mathfrak{f}) - \mathfrak{f}\|_{\mathrm{L_p}(\mathcal{D})} = \left(\int_\mathcal{D} |\mathcal{M}_m(\mathfrak{f}) - \mathfrak{f}|^p dz\right)^{\frac{1}{p}}$,
equation (6) allows us to write

$$\begin{aligned}
&\left\|\mathcal{H}_{m,\ell}(z) - z^\ell\right\|_{\mathrm{L_p}(\mathcal{D})} \\
&= \left\| x_m z[\mathcal{H}_{m,\ell-1}(z) - z^{\ell-1}]' + z[\mathcal{H}_{m,\ell-1}(z) - z^{\ell-1}] + x_m(\ell-1)z^{\ell-1} \right\|_{\mathrm{L_p}(\mathcal{D})} \\
&\leq 2^{\frac{1}{p}-1}\left[\left\| x_m z[\mathcal{H}_{m,\ell-1}(z) - z^{\ell-1}]' \right\|_{\mathrm{L_p}(\mathcal{D})} + \left\| z[\mathcal{H}_{m,\ell-1}(z) - z^{\ell-1}] \right\|_{\mathrm{L_p}(\mathcal{D})} + \left\| x_m(\ell-1)z^{\ell-1} \right\|_{\mathrm{L_p}(\mathcal{D})} \right] \\
&= 2^{\frac{1}{p}-1}\left[\left(\int_\mathcal{D} \left| x_m z[\mathcal{H}_{m,\ell-1}(z) - z^{\ell-1}]' \right|^p dz\right)^{\frac{1}{p}} + \left(\int_\mathcal{D} \left| z[\mathcal{H}_{m,\ell-1}(z) - z^{\ell-1}] \right|^p dz\right)^{\frac{1}{p}} + \left(\int_\mathcal{D} \left| x_m(\ell-1)z^{\ell-1} \right|^p dz\right)^{\frac{1}{p}} \right] \\
&\leq 2^{\frac{1}{p}-1}\left[x_m r\left(\int_\mathcal{D} \left| [\mathcal{H}_{m,\ell-1}(z) - z^{\ell-1}]' \right|^p dz\right)^{\frac{1}{p}} + \left(r\int_\mathcal{D} \left| [\mathcal{H}_{m,\ell-1}(z) - z^{\ell-1}] \right|^p dz\right)^{\frac{1}{p}} \right. \\
&\left. + x_m(\ell-1)\left(\int_\mathcal{D} \left| z^{\ell-1} \right|^p dz\right)^{\frac{1}{p}} \right],
\end{aligned}$$

hence

$$\left\|\mathcal{H}_{m,\ell}(z)-z^{\ell}\right\|_{L_p(\mathcal{D})}\leq 2^{\frac{1}{p}-1}\left[x_m r\left\|\left[\mathcal{H}_{m,\ell-1}(z)-z^{\ell-1}\right]'\right\|_{L_p(\mathcal{D})}\right.$$

$$\left.+r\left\|\mathcal{H}_{m,\ell-1}(z)-z^{\ell-1}\right\|_{L_p(\mathcal{D})}+x_m(\ell-1)\left\|z^{\ell-1}\right\|_{L_p(\mathcal{D})}\right].$$

We obtain the following from Equation (7) for $\mathcal{H}_{m,\ell}(z)$ polynomial of the degree ℓ -1 we obtain:

$$\left\|\left[\mathcal{H}_{m,\ell-1}(z)-z^{\ell-1}\right]'\right\|_{L_p(\mathcal{D})}\leq c\ell\left\|\mathcal{H}_{m,\ell-1}-w_{\ell-1}\right\|_{L_p(\mathcal{D})},$$

where c is a positive constant.

Therefore, it follows

$$\left\|\mathcal{H}_{m,\ell}(z)-z^{\ell}\right\|_{L_p(\mathcal{D})}\leq 2^{\frac{1}{p}-1}\left[c\ell x_m r\left\|\mathcal{H}_{m,\ell-1}-w_{\ell-1}\right\|_{L_p(\mathcal{D})}\right.$$

$$\left.+r\left\|\mathcal{H}_{m,\ell-1}-w_{\ell-1}\right\|_{L_p(\mathcal{D})}+x_m(\ell-1)\|w_{\ell-1}\|_{L_p(\mathcal{D})}\right],$$

consequently, we get

$$\left\|\mathcal{H}_{m,\ell}(z)-z^{\ell}\right\|_{L_p(\mathcal{D})}\leq 2^{\frac{1}{p}-1}\left[c\ell x_m r\left(\int_{\mathcal{D}}\left|\mathcal{H}_{m,\ell-1}-w_{\ell-1}\right|^p dz\right)^{\frac{1}{p}}\right.$$

$$\left.+r\left(\int_{\mathcal{D}}\left|\mathcal{H}_{m,\ell-1}-w_{\ell-1}\right|^p dz\right)^{\frac{1}{p}}+x_m(\ell-1)\left(\int_{\mathcal{D}}|w_{\ell-1}|^p dz\right)^{\frac{1}{p}}\right],$$

from (5) we get

$$\left\|\mathcal{H}_{m,\ell}(z)-z^{\ell}\right\|_{L_p(\mathcal{D})}\leq 2^{\frac{1}{p}-1}\left[c\ell x_m r\left(2(2r)^{\ell-1}(\pi r^2)^{\frac{1}{p}}\right)\right.$$

$$\left.+r\left(2(2r)^{\ell-1}(\pi r^2)^{\frac{1}{p}}\right)+x_m(\ell-1)r^{\ell-1}(\pi r^2)^{\frac{1}{p}}\right].$$

Thus, for any $\ell\geq 2$, we finally obtain

$$\left\|\mathcal{H}_{m,\ell}(z)-z^{\ell}\right\|_{L_p(\mathcal{D})}\leq x_m r(c+1)(2\pi r^2)^{\frac{1}{p}}\ell(2r)^{\ell-1}. \quad (8)$$

Now, using Lemma 3 we get

$$\mathcal{M}_m(\mathfrak{f})(z)=\sum_{\ell=0}^{\infty}a_\ell\mathcal{H}_{m,\ell}(z)$$

which implies

$$\|\mathcal{M}_m(\mathfrak{f})-\mathfrak{f}\|_{L_p(\mathcal{D})}=\left\|\sum_{\ell=0}^{\infty}a_\ell\mathcal{H}_{m,\ell}(z)-z^{\ell}\right\|_{L_p(\mathcal{D})}\leq\sum_{\ell=0}^{\infty}|a_\ell|\left\|\mathcal{H}_{m,\ell}(z)-z^{\ell}\right\|_{L_p(\mathcal{D})}$$

$$\leq x_m r\sum_{\ell=2}^{\infty}(c+1)(2\pi r^2)^{\frac{1}{p}}|a_\ell|\,\ell(2r)^{\ell-1},$$

since $\mathfrak{f}$ is analytic so

$\|\mathcal{M}_m(\mathfrak{f})-\mathfrak{f}\|_{L_p(\mathcal{D})}\leq x_m r\sum_{\ell=2}^{\infty}(c+1)(2\pi r^2)^{\frac{1}{p}}|a_\ell|\,\ell(2r)^{\ell-1}<\infty.$

The proof of Theorem 4 will make use of the following Voronovskaja type result.

Theorem 2: Assume that the hypothesis is same on $\mathfrak{f}$ and $\mathcal{R}$ in the statement of Theorem 1. If $1<r<\mathcal{R}/2$ is arbitrarily fixed, then for any $|z|\leq r$ any z and $m\in\mathbb{N}$, we have

$$\left\|\mathcal{M}_m(\mathfrak{f})(z)-\mathfrak{f}(z)-\frac{x_m}{2}z\mathfrak{f}''(z)\right\|_{L_p(\mathcal{D})}$$

$\leq (x_m)^2 10\mathrm{rc}(2\pi r^2)^{\frac{1}{p}} \sum_{\ell=2}^{\infty} |a_\ell| \ell(\ell-1)(\ell-2)^2(2r)^{\ell-3} < \infty.$

Proof: Using the recurrence relationship in (6), denoting

$$\mathcal{A}_{m,\ell}(z) = \mathcal{M}_m(w_\ell)(z) - w_\ell(z) - \frac{x_m}{2}\ell(\ell-1)z^{\ell-1}, \qquad (9)$$

we get

$\mathcal{A}_{m,\ell}(z) = z x_m \mathcal{A}'_{m,\ell-1}(z) + z\mathcal{A}_{m,\ell-1}(z) + \frac{(x_m)^2}{2} z^{\ell-2}(\ell-1)(\ell-2)^2.$

Hence

$$\left\|\mathcal{A}_{m,\ell}(z)\right\|_{L_p(\mathcal{D})} = \left\| z x_m \mathcal{A}'_{m,\ell-1}(z) + z\mathcal{A}_{m,\ell-1}(z) + \frac{(x_m)^2}{2} z^{\ell-2}(\ell-1)(\ell-2)^2 \right\|_{L_p(\mathcal{D})}$$

$$\leq 2^{\frac{1}{p}-1}\left[\left\| z x_m \mathcal{A}'_{m,\ell-1}(z)\right\|_{L_p(\mathcal{D})} + \left\| z\mathcal{A}_{m,\ell-1}(z)\right\|_{L_p(\mathcal{D})} + \left\|\frac{(x_m)^2}{2}(\ell-1)(\ell-2)^2 z^{\ell-2}\right\|_{L_p(\mathcal{D})}\right].$$

This suggests for any $|z| \leq r, m \in \mathbb{N}, \ell \geq 2$:

$$\left\|\mathcal{A}_{m,\ell}(z)\right\|_{L_p(\mathcal{D})} \leq 2^{\frac{1}{p}-1} r \left[x_m \left\|\mathcal{A}'_{m,\ell-1}(z)\right\|_{L_p(\mathcal{D})} + \frac{(x_m)^2}{2}(\ell-1)(\ell-2)^2 \left\| z^{\ell-3}\right\|_{L_p(\mathcal{D})}\right] + 2^{\frac{1}{p}-1} r \left\|\mathcal{A}_{m,\ell-1}(z)\right\|_{L_p(\mathcal{D})}.$$

Now from the Bernstein inequality we obtain

$\left\|\mathcal{A}'_{m,\ell-1}(z)\right\|_{L_p(\mathcal{D})} \leq c\ell \left\|\mathcal{A}_{m,\ell-1}(z)\right\|_{L_p(\mathcal{D})},$

where c is a positive constant.

From (9), we have

$$\left\|\mathcal{A}'_{m,\ell-1}(z)\right\|_{L_p(\mathcal{D})} \leq c\ell \left\|\mathcal{H}_{m,\ell-1}(z) - z^{\ell-1}\right\|_{L_p(\mathcal{D})} + c\ell \left\|\frac{x_m}{2}(\ell-1)(\ell-2) z^{\ell-2}\right\|_{L_p(\mathcal{D})}$$

$\leq 4 x_m c (\pi r^2)^{\frac{1}{p}} \ell(\ell-1)(\ell-2)(2r)^{\ell-2}.$

Hence

$$\left\|\mathcal{A}_{m,\ell}(z)\right\|_{L_p(\mathcal{D})} \leq 2^{\frac{1}{p}-1} r \left\|\mathcal{A}_{m,\ell-1}(z)\right\|_{L_p(\mathcal{D})} + 5r(x_m)^2 c(2\pi r^2)^{\frac{1}{p}} \ell(\ell-1)(\ell-2)^2(2r)^{\ell-3}.$$

Because it is the case that for $\ell = 1, 2$ we obtain $\mathcal{A}_{m,\ell}(z) = 0$, for $\ell \geq 3$ in the latter relation, we obtain that by using the same calculations as in [3] page 117, we obtain that

$\mathcal{A}_{m,\ell}(z) \leq 10r(x_m)^2 c(2\pi r^2)^{\frac{1}{p}} \ell(\ell-1)(\ell-2)^2(2r)^{\ell-3}.$

Since $\mathcal{M}_m(\mathfrak{f})(z) - \mathfrak{f}(z) - \frac{x_m}{2} z\mathfrak{f}''(z) \leq \sum_{\ell=0}^{\infty} a_\ell \mathcal{A}_{m,\ell}(z)$

$\left\|\mathcal{M}_m(\mathfrak{f})(z) - \mathfrak{f}(z) - \frac{x_m}{2} z\mathfrak{f}''(z)\right\|_{L_p(\mathcal{D})} = \left\|\sum_{\ell=0}^{\infty} a_\ell \mathcal{A}_{m,\ell}(z)\right\|_{L_p(\mathcal{D})} \leq \sum_{\ell=0}^{\infty} |a_\ell| \left\|\mathcal{A}_{m,\ell}(z)\right\|_{L_p(\mathcal{D})},$

we get the desired outcome by using the above inequality. ■

3. Results of approximation by using the operators $\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})$

In firstly, we prove an upper estimate in the simultaneous approximation by $\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})$.

Theorem 3: Assume that the statement of Theorem 1's function $\mathfrak{f}$ and constant $\mathcal{R}$ hypotheses are true.

(1) If $1 \le r < \mathcal{R}/2$ is arbitrary fixed and $0 \le \delta \le \gamma$, then for any $|z| \le r$ and $m \in \mathbb{N}$ we have

$$\left\| \mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z) - \mathfrak{f}(z) \right\|_{\mathrm{L_p}(\mathcal{D})}$$

$$\le \sum_{\ell=2}^{\infty} |a_\ell| r (2\pi r^2)^{\frac{1}{p}} (2r)^{\ell-1} + \sum_{\ell=2}^{\infty} |a_\ell| \frac{m x_m [(c+\gamma) r + \delta]}{2(m+\gamma)} (2\pi r^2)^{\frac{1}{p}} (2r)^{\ell-1}.$$

(2) Assume that $1 \le r < r_1 < \mathcal{R}/2$ and $0 \le \delta \le \gamma$. Then for any $|z| \le r$ and $q, m \in \mathbb{N}$ we have

$$\left\| \left[\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z) \right]^{(q)} - \mathfrak{f}^{(q)}(z) \right\|_{\mathrm{L_p}(\mathcal{D})}$$

$$\le \frac{q!\, r_1}{(r_1 - r)^{q+1}} \left[\sum_{\ell=2}^{\infty} |a_\ell| r (2\pi r^2)^{\frac{1}{p}} (2r)^{\ell-1} + \sum_{\ell=2}^{\infty} |a_\ell| \frac{m x_m [(c+\gamma) r + \delta]}{2(m+\gamma)} (2\pi r^2)^{\frac{1}{p}} (2r)^{\ell-1} \right].$$

Proof*:* (1) Clearly, Equation (4) allows us to write

$$\mathcal{H}_{m,\ell}^{(\delta,\gamma)}(z) - z^\ell = \frac{m x_m z}{m+\gamma} \left(\mathcal{H}_{m,\ell-1}^{(\delta,\gamma)}(z) \right)' + \left(\frac{m(\delta x_m + z)}{m+\gamma} \right) \mathcal{H}_{m,\ell-1}^{(\delta,\gamma)}(z) - z^\ell$$

$$= \frac{m x_m z}{m+\gamma} \left(\mathcal{H}_{m,\ell-1}^{(\delta,\gamma)}(z) \right)' + \left(\frac{m(\delta x_m + z)}{m+\gamma} \right) \left[\mathcal{H}_{m,\ell-1}^{(\delta,\gamma)}(z) - z^{\ell-1} \right] + \frac{\delta m x_m - \gamma z}{m+\gamma} z^{\ell-1}.$$

For all $z \in \mathbb{C}, \ell, m \in \mathbb{N}$.

$$\left\| \mathcal{H}_{m,\ell}^{(\delta,\gamma)}(z) - z^\ell \right\|_{\mathrm{L_p}(\mathcal{D})}$$

$$= \left\| \frac{m x_m z}{m+\gamma} \left(\mathcal{H}_{m,\ell-1}^{(\delta,\gamma)}(z) \right)' + \left(\frac{m(\delta x_m + z)}{m+\gamma} \right) \left[\mathcal{H}_{m,\ell-1}^{(\delta,\gamma)}(z) - z^{\ell-1} \right] + \frac{\delta m x_m - \gamma z}{m+\gamma} z^{\ell-1} \right\|_{\mathrm{L_p}(\mathcal{D})}.$$

By (5), we have the following from the above equation and the Bernstein inequality

$$\left\| \mathcal{H}_{m,\ell}^{(\delta,\gamma)}(z) - z^\ell \right\|_{\mathrm{L_p}(\mathcal{D})} \le 2^{\frac{1}{p}-1} \left[\frac{m x_m r}{m+\gamma} c\ell \left\| \mathcal{H}_{m,\ell-1}^{(\delta,\gamma)}(z) \right\|_{\mathrm{L_p}(\mathcal{D})} + \left(\frac{\delta m x_m}{m+\gamma} + r \right) \right.$$
$$\left. \left\| \mathcal{H}_{m,\ell-1}^{(\delta,\gamma)}(z) - z^{\ell-1} \right\|_{\mathrm{L_p}(\mathcal{D})} + \frac{(\delta+\gamma r) m x_m}{m+\gamma} \left\| z^{\ell-1} \right\|_{\mathrm{L_p}(\mathcal{D})} \right],$$

where c is a positive constant.
Hence, we get

$$\left\| \mathcal{H}_{m,\ell}^{(\delta,\gamma)}(z) - z^\ell \right\|_{\mathrm{L_p}(\mathcal{D})} \le 2^{\frac{1}{p}-1} [2r \left\| \mathcal{H}_{m,\ell-1}^{(\delta,\gamma)}(z) - z^{\ell-1} \right\|_{\mathrm{L_p}(\mathcal{D})}$$
$$+ \frac{m x_m r}{m+\gamma} c\ell \left\| \mathcal{H}_{m,\ell-1}^{(\delta,\gamma)}(z) \right\|_{\mathrm{L_p}(\mathcal{D})} + \frac{(\delta+\gamma r) m x_m}{m+\gamma} \left\| z^{\ell-1} \right\|_{\mathrm{L_p}(\mathcal{D})} \Bigg].$$

Now by using definition $\mathrm{L_p}(\mathcal{D})$ for any $\ell \ge 2$, we get

$$\left\| \mathcal{H}_{m,\ell}^{(\delta,\gamma)}(z) - z^\ell \right\|_{\mathrm{L_p}(\mathcal{D})} \le r (2\pi r^2)^{\frac{1}{p}} (2r)^{\ell-1} + \frac{m x_m [(c+\gamma) r + \delta]}{2(m+\gamma)} (2\pi r^2)^{\frac{1}{p}} (2r)^{\ell-1}. \tag{10}$$

For the complex Szasz operators, it is known that
$\Psi_m(\mathfrak{f})(z) = \sum_{\ell=0}^{\infty} a_\ell \Psi_m(w_\ell)(z) = \sum_{\ell=0}^{\infty} a_\ell \mathcal{H}_m(z)$ for all $|z| \le r$,
we can write using the same idea in [7], pages 116–117

$$\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z) = \sum_{\ell=0}^{\infty} a_\ell \mathcal{M}_m^{(\delta,\gamma)}(w_\ell)(z) = \sum_{\ell=0}^{\infty} a_\ell \mathcal{H}_m^{(\delta,\gamma)}(z).$$

Hence by (10) we obtain

$$\left\|\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z)-\mathfrak{f}(z)\right\|_{\mathrm{L_p}(\mathcal{D})} \le \sum_{\ell=0}^{\infty}|a_\ell|\left\|\mathcal{H}_m^{(\delta,\gamma)}(z)-z^\ell\right\|_{\mathrm{L_p}(\mathcal{D})}$$

$\le \sum_{\ell=2}^{\infty}|a_\ell| r(2\pi r^2)^{\frac{1}{p}}(2r)^{\ell-1}+\sum_{\ell=2}^{\infty}|a_\ell|\frac{mx_m[(c+\gamma)r+\delta]}{2(m+\gamma)}(2\pi r^2)^{\frac{1}{p}}(2r)^{\ell-1}.$

Since by hypothesis, $\mathfrak{f}(z)=\sum_{\ell=0}^{\infty}a_\ell z^\ell$ is uniformly and absolutely convergent in $|z|\le r$, for any $1\le r<\mathcal{R}/2$, it is clear that $\sum_{\ell=2}^{\infty}|a_\ell|(2r)^{\ell-1}<\infty$.

(2) Implying by K the circle with radius $r_1>r$ and its center at zero, for any $|z|\le r$ and $u\in$ K we obtain $|u-z|\ge r_1-r$, and using Cauchy's formula, it follows for all $|z|\le r$

$$\left\|\left[\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z)\right]^q-\mathfrak{f}^q(z)\right\|_{\mathrm{L_p}(\mathcal{D})}=\frac{q!}{2\pi}\left\|\int_{\mathrm{K}}\frac{\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z)-\mathfrak{f}(z)}{(u-z)^{q+1}}dz\right\|_{\mathrm{L_p}(\mathcal{D})}$$

$$\le\frac{q!\,r_1}{(r_1-r)^{q+1}}\left[\sum_{\ell=2}^{\infty}|a_\ell| r(2\pi r^2)^{\frac{1}{p}}(2r)^{\ell-1}+\sum_{\ell=2}^{\infty}|a_\ell|\frac{mx_m[(c+\gamma)r+\delta]}{2(m+\gamma)}(2\pi r^2)^{\frac{1}{p}}(2r)^{\ell-1}\right],$$

which proves (2) and the theorem.■

We provide the following: Voronovsskaja Formu1a for $\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})$.

Theorem 4*:* Assume that the $\mathfrak{f}$ and $\mathcal{R}$ hypotheses in the proof of Theorem1 are true**.** The Voronovskaja type result then holds for any $|z|\le r$ with $1\le r<\mathcal{R}/2$ *and* $m\in\mathbb{N}$:

$$\left\|\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z)-\mathfrak{f}(z)-\frac{\delta m x_m-\gamma z}{m+\gamma}\mathfrak{f}'(z)-\frac{m x_m z}{2m}\mathfrak{f}''(z)\right\|_{\mathrm{L_p}(\mathcal{D})}$$

$$\le (x_m)^2 V_1(\mathfrak{f})+2^{\frac{1}{p}-1}\left(\frac{m x_m}{m+\gamma}\right)^2(\pi r^2)^{\frac{1}{p}}\sum_{\zeta=2}^{6}V_\zeta(\mathfrak{f}),$$

where $V_1(\mathfrak{f})=10\mathrm{rc}(2\pi r^2)^{\frac{1}{p}}\sum_{\ell=2}^{\infty}|a_\ell|\ell(\ell-1)(\ell-2)^2(2r)^{\ell-3}<\infty$,

$$V_2(\mathfrak{f})=\delta^2\sum_{\ell=2}^{\infty}|a_\ell|\,\ell(\ell-1)(2r)^{\ell-2}<\infty,$$

$$V_3(\mathfrak{f})=\delta r\sum_{\ell=2}^{\infty}|a_\ell|\ell(\ell-1)(\ell-2)(2r)^{\ell-3}<\infty,$$

$$V_4(\mathfrak{f})=(r\gamma^2+\gamma)\sum_{\ell=2}^{\infty}|a_\ell|\,\ell^2(\ell-1)(2r)^{\ell-2}<\infty,$$

$$V_5(\mathfrak{f})=(\gamma r)^2\sum_{\ell=2}^{\infty}|a_\ell|\ell(\ell-1)r^{\ell-2}<\infty,$$

$$V_6(\mathfrak{f})=\delta\gamma r\sum_{\ell=2}^{\infty}|a_\ell|\ell(\ell-1)r^{\ell-2}<\infty.$$

Proof*:* Let's consider for all $z\in\mathcal{D}_{\mathcal{R}}$

$\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z)-\mathfrak{f}(z)-\dfrac{\delta m x_m-\gamma z}{m+\gamma}\mathfrak{f}'(z)-\dfrac{m x_m z}{2m}\mathfrak{f}''(z)$

$=\mathcal{M}_m(\mathfrak{f})(z)-\mathfrak{f}(z)-\frac{m x_m z}{2m}\mathfrak{f}''(z)+\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z)-\mathcal{M}_m(\mathfrak{f})(z)-\frac{\delta m x_m-\gamma z}{m+\gamma}\mathfrak{f}'(z).$

Since $\mathfrak{f}(z)=\sum_{\ell=0}^{\infty}a_\ell z^\ell$, we obtain

$$\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z) - \mathfrak{f}(z) - \frac{\delta m x_m - \gamma z}{m+\gamma}\mathfrak{f}'(z) - \frac{m x_m z}{2m}\mathfrak{f}''(z)$$
$$= \sum_{\ell=2}^{\infty} a_\ell \left(\mathcal{M}_m(w_\ell)(z) - z^\ell - \frac{m x_m z}{2m}\ell(\ell-1)z^{\ell-2}\right)$$
$$+ \sum_{\ell=2}^{\infty} a_\ell \left(\mathcal{M}_m^{(\delta,\gamma)}(w_\ell)(z) - \mathcal{M}_m(w_\ell)(z) - \frac{\delta m x_m - \gamma z}{m+\gamma}\ell z^{\ell-1}\right).$$

First, the Voronoskaja type result for the $\mathcal{M}_m(\mathfrak{f})(z)$ operetors is applied, which can be found in Theorem 2. Then the first sum is
$\left\| \mathcal{M}_m(\mathfrak{f})(z) - \mathfrak{f}(z) - \frac{x_m}{2} z\mathfrak{f}''(z) \right\|_{L_p(\mathcal{D})} \leq (x_m)^2 10 \mathrm{rc}(2\pi r^2)^{\frac{1}{p}} \sum_{\ell=2}^{\infty} |a_\ell| \ell(\ell-1)(\ell-2)^2 (2r)^{\ell-3}$.
We now estimate the second sum. Since $\mathcal{M}_m^{(\delta,\gamma)}(w_\ell)(z) = \mathcal{H}_{m,\ell}^{(\delta,\gamma)}(z)$ and considering that $\mathcal{M}_m^{(\delta,\gamma)}(w_\ell)(z) = \sum_{\zeta=0}^{\ell} \binom{\ell}{\zeta} \frac{m^\zeta(\delta m x_m)^{\ell-\zeta}}{(m+\gamma)^\ell} \mathcal{M}_m(w_\zeta)(z)$ with$\mathcal{M}_m^{(0,0)}(w_\ell)(z) = \mathcal{M}_m(w_\zeta)(z)$ for $\ell, m \in \mathbb{N} \cup \{0\}$ we can write
$$\mathcal{M}_m^{(\delta,\gamma)}(w_\ell)(z) - \mathcal{M}_m(w_\ell)(z) - \frac{\delta m x_m - \gamma z}{m+\gamma}\ell z^{\ell-1}$$
$$= \mathcal{H}_{m,\ell}^{(\delta,\gamma)}(z) - \mathcal{H}_{m,\ell}(z) - \frac{\delta m x_m - \gamma z}{m+\gamma}\ell z^{\ell-1}$$
$$= \sum_{\zeta=0}^{\ell-1} \binom{\ell}{\zeta} \frac{m^\zeta(\delta m x_m)^{\ell-\zeta}}{(m+\gamma)^\ell}\mathcal{H}_{m,\zeta}(z) + \left(\frac{m^\ell}{(m+\gamma)^\ell} - 1\right)\mathcal{H}_{m,\ell}(z) - \frac{\delta m x_m - \gamma z}{m+\gamma}\ell z^{\ell-1}$$
$$= \sum_{\zeta=0}^{\ell-2} \binom{\ell}{\zeta} \frac{m^\zeta(\delta m x_m)^{\ell-\zeta}}{(m+\gamma)^\ell}\mathcal{H}_{m,\zeta}(z) + \frac{\ell m^{\ell-1}\delta m x_m}{(m+\gamma)^\ell}\mathcal{H}_{m,\ell-1}(z) -$$
$$-\sum_{\zeta=0}^{\ell-1} \binom{\ell}{\zeta} \frac{m^\zeta(\gamma)^{\ell-\zeta}}{(m+\gamma)^\ell}\mathcal{H}_{m,\ell}(z) - \frac{\delta m x_m - \gamma z}{m+\gamma}\ell z^{\ell-1}$$
$$= \sum_{\zeta=0}^{\ell-2} \binom{\ell}{\zeta} \frac{m^\zeta(\delta m x_m)^{\ell-\zeta}}{(m+\gamma)^\ell}\mathcal{H}_{m,\zeta}(z) + \frac{\ell m^{\ell-1}\delta m x_m}{(m+\gamma)^\ell}\left[\mathcal{H}_{m,\ell-1}(z) - z^{\ell-1}\right] -$$
$$-\sum_{\zeta=0}^{\ell-2} \binom{\ell}{\zeta} \frac{m^\zeta(\gamma)^{\ell-\zeta}}{(m+\gamma)^\ell}\mathcal{H}_{m,\ell}(z) - \frac{\ell m^{\ell-1}\gamma}{(m+\gamma)^\ell}\left[\mathcal{H}_{m,\ell}(z) - z^\ell\right]$$
$$+ \frac{\ell\delta m x_m}{(m+\gamma)}\left(\frac{m^{\ell-1}}{(m+\gamma)^{\ell-1}} - 1\right)z^{\ell-1} + \frac{\ell\gamma}{(m+\gamma)}\left(1 - \frac{m^{\ell-1}}{(m+\gamma)^{\ell-1}}\right)z^\ell.$$
Considering the condition (2), we obtain
$$\sum_{\zeta=0}^{\ell-2} \binom{\ell-2}{\zeta} \frac{m^\zeta(\delta m x_m)^{\ell-2-\zeta}}{(m+\gamma)^{\ell-2}} = \sum_{\zeta=0}^{\ell-2} \binom{\ell-2}{\zeta} \frac{m^\zeta}{(m+\gamma)^\zeta} \frac{(\delta m x_m)^{\ell-2-\zeta}}{(m+\gamma)^{\ell-2-\zeta}}$$
$$= \left(\frac{m+\delta m x_m}{m+\gamma}\right)^{\ell-2}.$$
Moreover, we have the following inequities:
$$1 - \frac{m^\ell}{(m+\gamma)^\ell} = 1 - \prod_{\zeta=0}^{\ell} \frac{m}{m+\gamma} \leq \sum_{\zeta=1}^{\ell}\left(1 - \frac{m}{m+\gamma}\right) = \frac{\ell\gamma}{m+\gamma},$$

and from (5) $\left\|\mathcal{H}_{m,\ell}(z)\right\|_{L_p(\mathcal{D})} \leq (2r)^{\ell}(\pi r^2)^{\frac{1}{p}}$. And hence, we get

$$\left\|\sum_{\zeta=0}^{\ell-2}\binom{\ell}{\zeta}\frac{m^{\zeta}(\delta m x_m)^{\ell-\zeta}}{(m+\gamma)^{\ell}}\mathcal{H}_{m,\zeta}(z)\right\|_{L_p(\mathcal{D})} \leq \sum_{\zeta=0}^{\ell-2}\binom{\ell}{\zeta}\frac{m^{\zeta}(\delta m x_m)^{\ell-\zeta}}{(m+\gamma)^{\ell}}\left\|\mathcal{H}_{m,\zeta}(z)\right\|_{L_p(\mathcal{D})}$$

$$\leq \sum_{\zeta=0}^{\ell-2}\frac{\ell(\ell-1)}{(\ell-\zeta)(\ell-\zeta-1)}\binom{\ell-2}{\zeta}\frac{m^{\zeta}(\delta m x_m)^{\ell-\zeta}}{(m+\gamma)^{\ell}}\left\|\mathcal{H}_{m,\zeta}(z)\right\|_{L_p(\mathcal{D})}$$

$$\leq \ell(\ell-1)\frac{(\delta m x_m)^2}{(m+\gamma)^2}(2r)^{\ell-2}(\pi r^2)^{\frac{1}{p}}$$

As a result, using (8):

$$\left\|\mathcal{H}_{m,\ell}^{(\delta,\gamma)}(z)-\mathcal{H}_{m,\ell}(z)-\frac{\delta m x_m-\gamma z}{m+\gamma}\ell z^{\ell-1}\right\|_{L_p(\mathcal{D})}$$

$$\leq 2^{\frac{1}{p}-1}\left[\left\|\sum_{\zeta=0}^{\ell-2}\binom{\ell}{\zeta}\frac{m^{\zeta}(\delta m x_m)^{\ell-\zeta}}{(m+\gamma)^{\ell}}\mathcal{H}_{m,\zeta}(z)\right\|_{L_p(\mathcal{D})}+\frac{\ell m^{\ell-1}\delta m x_m}{(m+\gamma)^{\ell}}\left\|\mathcal{H}_{m,\ell-1}(z)-z^{\ell-1}\right\|_{L_p(\mathcal{D})}\right.$$

$$+\sum_{\zeta=0}^{\ell-2}\binom{\ell}{\zeta}\frac{m^{\zeta}(\gamma)^{\ell-\zeta}}{(m+\gamma)^{\ell}}\left\|\mathcal{H}_{m,\ell}(z)\right\|_{L_p(\mathcal{D})}-\frac{\ell m^{\ell-1}\gamma}{(m+\gamma)^{\ell}}\left\|\mathcal{H}_{m,\ell}(z)-z^{\ell}\right\|_{L_p(\mathcal{D})}+$$

$$\left.+\frac{\ell\delta m x_m}{(m+\gamma)}\left(\frac{m^{\ell-1}}{(m+\gamma)^{\ell-1}}-1\right)\left\|z^{\ell-1}\right\|_{L_p(\mathcal{D})}+\frac{\ell\gamma}{(m+\gamma)}\left(1-\frac{m^{\ell-1}}{(m+\gamma)^{\ell-1}}\right)\left\|z^{\ell}\right\|_{L_p(\mathcal{D})}\right]$$

$$\leq 2^{\frac{1}{p}-1}\left[\ell(\ell-1)\frac{(\delta m x_m)^2}{(m+\gamma)^2}(\pi r^2)^{\frac{1}{p}}(2r)^{\ell-2}\right.$$

$$+\frac{\ell m^{\ell-1}\delta m x_m}{(m+\gamma)^{\ell}}x_m r(\ell-1)(\ell-2)(\pi r^2)^{\frac{1}{p}}(2r)^{\ell-3}$$

$$+\frac{\ell(\ell-1)\gamma^2}{(m+\gamma)^2}(\pi r^2)^{\frac{1}{p}}(2r)^{\ell}+\frac{\ell m^{\ell-1}\gamma}{(m+\gamma)^{\ell}}x_m r\ell(\ell-1)(\pi r^2)^{\frac{1}{p}}(2r)^{\ell-2}$$

$$\left.+\frac{\ell(\ell-1)\delta\gamma m x_m}{(m+\gamma)^2}(\pi r^2)^{\frac{1}{p}}r^{\ell-1}+\frac{\ell(\ell-1)\gamma^2}{(m+\gamma)^2}(\pi r^2)^{\frac{1}{p}}r^{\ell}\right]$$

$$\leq 2^{\frac{1}{p}-1}\left[\delta^2\left(\frac{m x_m}{m+\gamma}\right)^2(\pi r^2)^{\frac{1}{p}}\ell(\ell-1)(2r)^{\ell-2}+\delta r\left(\frac{m x_m}{m+\gamma}\right)^2(\pi r^2)^{\frac{1}{p}}\ell(\ell-1)(\ell-2)(2r)^{\ell-3}\right.$$

$$+\left(\frac{m x_m}{m+\gamma}\right)^2(\pi r^2)^{\frac{1}{p}}(r\gamma^2+\gamma)\ell^2(\ell-1)(2r)^{\ell-1}$$

$$\left.+\delta\gamma\left(\frac{m x_m}{m+\gamma}\right)^2(\pi r^2)^{\frac{1}{p}}\ell(\ell-1)r^{\ell-1}+\gamma^2\left(\frac{m x_m}{m+\gamma}\right)^2(\pi r^2)^{\frac{1}{p}}\ell(\ell-1)r^{\ell}\right],$$

this complete the proof of the theorem. ■

Now, the result that follows will be helpful in determining the precise degree of approximation $\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})$.

Theorem 5*:* Assume that $\mathfrak{f}$ is not a polynomial of degree ≤ 0 and that the hypotheses on $\mathfrak{f}$ and $\mathcal{R}$ are the same. Following that, we have for all $|z| \leq r$ and $m \in \mathbb{N}$:

$$\left\|\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z)-\mathfrak{f}(z)\right\|_{L_p(\mathcal{D})} \geq x_m\, V_{L_p(\mathcal{D})}(\mathfrak{f}),$$

with that as the constant $V_{L_p(\mathcal{D})}(\mathfrak{f})$ depends only on $\mathfrak{f}$ and $L_p(\mathcal{D})$.

Proof: We can write for any $|z| \le r$ and $m \in \mathbb{N}$

$$\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z) - \mathfrak{f}(z) = x_m \left[\frac{1}{x_m}\frac{(\delta m x_m - \gamma z)}{m+\gamma}\mathfrak{f}'(z) + \frac{z}{2}\mathfrak{f}''(z) + x_m\left(\frac{1}{x_m}\right)^2\right.$$

$$\left.\left(\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z) - \mathfrak{f}(z) - \frac{\delta m x_m - \gamma z}{m+\gamma}\mathfrak{f}'(z) - \frac{z x_m}{2}\mathfrak{f}''(z) - x_m \frac{\gamma(\delta m x_m - \gamma z)\mathfrak{f}'(z)}{(m+\gamma)}\right)\right].$$

Using the equality $\|Z_1 + Z_2\|_{L_p(\mathcal{D})} \ge \left|\|Z_1\|_{L_p(\mathcal{D})} - \|Z_2\|_{L_p(\mathcal{D})}\right| \ge \|Z_1\|_{L_p(\mathcal{D})} - \|Z_2\|_{L_p(\mathcal{D})}$ we get

$$\left\|\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f})(z) - \mathfrak{f}(z)\right\|_{L_p(\mathcal{D})} \ge x_m\left[(\delta m x_m - \gamma w_1)\mathfrak{f}' + \frac{w_1}{2}\mathfrak{f}''\right.$$

$$\left.-x_m\left(\frac{1}{x_m}\right)^2\left\|\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f}) - \mathfrak{f} - \frac{\delta m x_m - \gamma w_1}{m+\gamma}\mathfrak{f}' - \frac{x_m}{2}w_1\mathfrak{f}'' - x_m\frac{\gamma(\delta m x_m - \gamma w_1)}{(m+\gamma)}\mathfrak{f}'\right\|_{L_p(\mathcal{D})}\right].$$

Since $\mathfrak{f}$ is not a polynomial with degree≤ 0 in $\mathcal{D}_{\mathcal{R}}$, so we get

$$\left\|(\delta m x_m - \gamma w_1)\mathfrak{f}' + \frac{w_1}{2}\mathfrak{f}''\right\|_{L_p(\mathcal{D})} \ge \left\|(\delta m x_1 - \gamma w_1)\mathfrak{f}' + \frac{w_1}{2}\mathfrak{f}''\right\|_{L_p(\mathcal{D})} > 0.$$

In fact, assuming the contrary, it follows that for all $\in \overline{\mathcal{D}_{\mathcal{R}}}$ $(\delta m x_1 - \gamma z)\mathfrak{f}'(z) + \frac{z}{2}\mathfrak{f}''(z) = 0$. By implying $g(z) = \mathfrak{f}(z)$, staring for $g(z)$ in the form $g(z) = \sum_{\ell=0}^{\infty} K_\ell z^\ell$ and simply replacing in the previous differential equation, we can use method as in [6] or [7] p.75-76 we get $K_\ell = 0$ for all $\ell = 0, 1$. As a result, we find that $\mathfrak{f}(z)$ is a constant function, which is a contradiction.

Using Theorem 4, we get

$$\left(\frac{1}{x_m}\right)^2\left\|\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f}) - \mathfrak{f} - \frac{\delta m x_m - \gamma w_1}{m+\gamma}\mathfrak{f}' - \frac{x_m}{2}w_1\mathfrak{f}'' - x_m\frac{\gamma(\delta m x_m - \gamma w_1)}{(m+\gamma)}\mathfrak{f}'\right\|_{L_p(\mathcal{D})}$$

$$\le \left(\frac{1}{x_m}\right)^2\left\|\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f}) - \mathfrak{f} - \frac{\delta m x_m - \gamma w_1}{m+\gamma}\mathfrak{f}' - \frac{x_m}{2}w_1\mathfrak{f}''\right\|_{L_p(\mathcal{D})} + \|\gamma(\delta m x_m - \gamma w_1)\mathfrak{f}'\|_{L_p(\mathcal{D})}$$

$$\le \sum_{\zeta=2}^{6} V_{\zeta, L_p(\mathcal{D})}(\mathfrak{f}) + \gamma(\delta m x_m - \gamma w_1)\|\mathfrak{f}'\|_{L_p(\mathcal{D})},$$

there is $m_1 > m_0$ such that for all $m > m_1$ we have (only depending on $\mathfrak{f}, \delta, \gamma$ *and* $L_p(\mathcal{D})$)

$$\left\|(\delta m x_m - \gamma w_1)\mathfrak{f}' + \frac{w_1}{2}\mathfrak{f}''\right\|_{L_p(\mathcal{D})}$$

$$-x_m\left(\frac{1}{x_m}\right)^2\left\|\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f}) - \mathfrak{f} - \frac{\delta m x_m - \gamma w_1}{m+\gamma}\mathfrak{f}' - \frac{x_m}{2}w_1\mathfrak{f}''\right\|_{L_p(\mathcal{D})}$$

$$\ge \frac{1}{2}\left\|(\delta m x_m - \gamma w_1)\mathfrak{f}' + \frac{w_1}{2}\mathfrak{f}''\right\|_{L_p(\mathcal{D})},$$

which implies

$$\left\|\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f}) - \mathfrak{f}\right\|_{L_p(\mathcal{D})} \ge \frac{x_m}{2}\left\|(\delta m x_m - \gamma w_1)\mathfrak{f}' + \frac{w_1}{2}\mathfrak{f}''\right\|_{L_p(\mathcal{D})}.$$

For all $m > m_1$. For $m \in \{m_0 + 1, \dots, m_1\}$, we obtain $\left\|\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f}) - \mathfrak{f}\right\|_{L_p(\mathcal{D})} \ge x_m U_{L_p}(\mathfrak{f})$ with $U_{L_p(\mathcal{D})}(\mathfrak{f}) = \frac{1}{x_m}\left\|\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f}) - \mathfrak{f}\right\|_{L_p(\mathcal{D})}$ which implies $\left\|\mathcal{M}_m^{(\delta,\gamma)}(\mathfrak{f}) - \mathfrak{f}\right\|_{L_p(\mathcal{D})} \ge V_{L_p(\mathcal{D}),\mathfrak{f}}\, x_m$ for all $m > m_0$, with

$$V_{L_p(\mathcal{D}),f} = min\left\{U_{L_p,m_0+1}(f), \dots, U_{L_p(\mathcal{D}),m_1}(f), \frac{1}{2}\left\|(\delta m x_m - \gamma w_1)f' + \frac{w_1}{2}f''\right\|_{L_p(\mathcal{D})}\right\},$$

this completes the theorem's proof. ■

We are now ready to state the exact approximation degree for $\mathcal{M}_m^{(\delta,\gamma)}(f)$.

4. Conclusions

1- Theorems 5 and 3(1) make it clear that if f is not a constant function, then the exact degree in the operator $\mathcal{M}_m^{(\delta,\gamma)}(f)$ approximation is x_m.

2- The exact degree in the simultaneous approximation by using the operator $\left[\mathcal{M}_m^{(\delta,\gamma)}(f)(z)\right]^{(q)}$ is shown to be x_m by taking into account Theorem 3 (2) and performing calculations in a manner similar to this one as in [7] p. 119 for the case of classical complex Szasz-Mirakjian operators.

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