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On Goldie lifting modules

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Abstract

Let R be an associative ring with identity and let M be a unital left R- module. We introduced the following concept .An R- module M is called a Goldie lifting module (briefly $\mathcal{G}$-lifting) if for every proper submodule*A* of *M*, there exists a proper direct summand *D* of *M* such that $D \leq_{ce} A+D$ in *M*. The main purpose of this work is to define goldie lifting modules and we give examples and basic properties of $\mathcal{G}$-lifting modules.

Keywords: Goldie lifting modules.

حولمقاسات الرفع من النمط غولدي

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الخلاصة:

لتكن R حلقة تجميعية ذات عنصر محايد و M مقاس احادي ايسر معرف عليها. قدمنا مفهوم مقاس الرفع من النمط _ $\mathcal{G}$ يدعى المقاس M مقاس رفع من النمط _ $\mathcal{G}$اذا كان لكل مقاس جزئي فعلي A من M يوجد مركبة مجموع مباشر D من M بحيث ان $D \leq_{ce} A+D$ في M . الغرض الرئيسي من هذا البحث هو تعريف مقاس الرفع من النمط _ $\mathcal{G}$ و اعطاء امثله والخواص الاساسيه لهذا النوع من المقاسات.

Introduction:

Recall that a submodule*A* of an *R*- module *M* is called a small submodule of *M* (denoted by $A \ll M$) , if for any submodule*B* of *M* such that $M=A+B$, then $M=B$, see [1-3].Let *M* be an *R*- module and let *A*, *B* be submodules of *M* such that $A \leq B \leq M$. Recall that *A* is called a coessentialsubmodule of *B* in *M* (briefly $A \leq_{ce} B$ in *M*) if $\frac{B}{A} \ll \frac{M}{A}$, see [1].

Let *M* be an *R*- module. Recall that a submodule*B* of *M* is called a coclosedsubmodule of *M* (notation $B \leq_{cc} M$), if $\frac{B}{A} \ll \frac{M}{A}$ implies that $A=B$, $\forall A \leq B$, see [1].

Following [4], the following is a relation on a set of submodules of *M*.$A \; \beta \; B$ if $A \cap B \leq_e A$ and $A \cap B \leq_e B$. An *R*- module *M* is called an $\mathcal{G}$-extending module if for every submodule*A* of *M* , there exists a direct summand *D* of *M* such that $(A \; \beta \; D)$, see [4].

An *R*- module *M* is called a lifting module if for every submodule*A* of *M* , there exists a submodule*D* of *A* such that $M=D \oplus D'$ and $A \cap D' \ll D'$, see [5].

These observations lead us to introduce the following relations on the set of submodules of *M*.

$A \, \gamma \, B$ if $B \leq_{ce} A+B$ in *M*.

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We define M to be Goldie lifting module (briefly $\mathcal{G}$-lifting) if for every proper submodule A of M, there exists a proper direct summand D of M such that $A\,\gamma D$.

This paper consists of two sections

In section one , we introduced the concept of γ with some examples and basic properties.

In section two , we define a goldie lifting modules also we give some characterization of goldie lifting modules and investigate various conditions for a direct sum of $\mathcal{G}$-lifting modules to be $\mathcal{G}$-lifting.

§1: Basic properties of γ

In this section , we define a relation γ on the set of submodules of a module M and we will illustrate it by some examples. We also give some basic properties . We start by a definition.

Definition (1.1): Let M be an R- module and let γ be a relation on the set of submodules of M defined as follows: $\boldsymbol{A\gamma B}$ if $B \leq_{ce} A+B$ in M.

Remarks and Examples (1.2)

1- In Z as Z- module. Let $A=6Z$ and $B=4Z$. One can easily show that $B \leq_{ce} A+B=2Z$ in Z.

2- Let A and B be submodules of an R- module M such that $B\leq A$, then $A\,\gamma B$ if and only if $B\leq_{ce}A$ in M. For example Z_8 as Z- module. It is easy to see that $\{\bar{0},\bar{2},\bar{4},\bar{6}\}\,\gamma\,\{\bar{0},\bar{4}\}$, where $\{\bar{0},\bar{4}\}\leq_{ce}\{\bar{0},\bar{2},\bar{4},\bar{6}\}$ in Z_8.

3- Let A be a submodule of an R- module M. One can easily show that $A\,\gamma\, 0$ if and only if $A\ll M$.

4- Let M be an R- module . Clearly that $A\,\gamma M$, for every submodule A of M.

Proposition (1.3): γ is reflexive and transitive relation.

Proof:Clearly that γ is reflexive.To show γ is transitive , let A , B and C be submodules of a module M such that $A\,\gamma B$ and $B\,\gamma C$, then $\frac{A+B}{B}\ll\frac{M}{B}$ and $\frac{B+C}{C}\ll\frac{M}{C}$. We have to show that $A\,\gamma C$ i.e, $\frac{A+C}{C}\ll\frac{M}{C}$. Let U be submodule of M containing C such that $\frac{M}{C}=\frac{A+C}{C}+\frac{U}{C}$, then $M=A+B+C+U$ and hence $\frac{M}{B}=\frac{A+B}{B}+\frac{B+C}{B}+\frac{U+B}{B}$. But $\frac{A+B}{B}\ll\frac{M}{B}$, therefore $\frac{M}{B}=\frac{B+C}{B}+\frac{U+B}{B}$, hence $M=B+C+U$. Now $\frac{M}{C}=\frac{U}{C}+\frac{B+C}{C}$ .Since $\frac{B+C}{C}\ll\frac{M}{C}$, then $\frac{M}{C}=\frac{U}{C}$ and hence $M=U$.Thus $\frac{A+C}{C}\ll\frac{M}{C}$.

Note: In general , γ is not symmetric. For example , consider Z as Z- module. Clearly $2Z\,\gamma Z$. But $2Z$ is not coessential submodule of Z in Z. Thus Z is not related with $2Z$ by γ.

The following proposition gives a characterization of γ.

Proposition (1.4): Let A and B be submodules of an R- module M .Then $A\,\gamma B$ if and only if $M= A+X$ implies $M=B+X$, for each submodule X of M.

Proof: ($\Rightarrow$) Suppose that $A\,\gamma B$ and let X be a submodule of M such that $M= A+X$, then $\frac{M}{B}=\frac{A+B}{B}+\frac{X+B}{B}$. But $A\,\gamma B$, therefore $\frac{A+B}{B}\ll\frac{M}{B}$ and hence $\frac{M}{B}=\frac{X+B}{B}$. Thus $M=B+X$.

($\Leftarrow$) To show $A\,\gamma B$. Let $\frac{M}{B}=\frac{A+B}{B}+\frac{U}{B}$, where $\frac{U}{B}\leq\frac{M}{B}$. Then $M= A+B+U$. By our assumption , we get $M = B+U$. But $B\leq U$, therefore $M = U$. Thus $A\,\gamma B$.

Proposition (1.5): Let M be an R- module and let A , B and C submodules of M.

1- If $A\,\gamma B$ and $B\ll M$, then $A\ll M$.

2- If $C \ll M$ and $A \leq B+C$. Then $A\,\gamma\,B$.

Proof:

1- Let U be a submodule of M such that $M = A+U = A+B+U = B+U$, by prop. (1.4). But $B \ll M$, therefore $M = U$. Thus $A \ll M$.

2- To show $A\,\gamma\,B$. Let $M = A+X$, where $X \leq M$, then $M = A+B+C+X = B+C+X$, since $A \leq B+C$. But $C \ll M$, therefore $M = B+X$. Thus $A\,\gamma\,B$, by prop.(1.4).

Proposition (1.6): Let M be an R- module and let A , $B \leq M$, then $A\,\gamma\,B$ if and only if $\frac{A}{L}\gamma\frac{B}{L}$, for every submodule L contained in A and B.

Proof: Let $A\,\gamma\,B$ and let $L \leq M$ contained in A and B , then $B \leq_{ce} A+B$ in M. By [1] ,$\frac{B}{L} \leq_{ce} \frac{A+B}{L} = \frac{A}{L} + \frac{B}{L}$ in $\frac{M}{L}$. Thus$\frac{A}{L}\gamma\frac{B}{L}$.

For the converse , suppose that $\frac{A}{L}\gamma\frac{B}{L}$for every submodule L contained in A and B, then $\frac{B}{L} \leq_{ce} \frac{A}{L} + \frac{B}{L} = \frac{A+B}{L}$ in $\frac{M}{L}$. By [1], we get $B \leq_{ce} A+B$ in M. Thus $A\,\gamma\,B$

Proposition (1.7): Let A_1, A_2 ,B_1 and B_2 be submodules of an R- module M such that $A_1\gamma B_1$ and $A_2\gamma B_2$, then $(A_1+A_2)\,\gamma\,(B_1+B_2)$.

Proof: Assume that $A_1\gamma B_1$ and $A_2\gamma B_2$. Then $B_1 \leq_{ce} A_1+B_1$ in M and $B_2 \leq_{ce} A_2+B_2$ in M. So $B_1+B_2 \leq_{ce} A_1+A_2+B_1+B_2$ in M , by [1]. Thus $(A_1+A_2)\,\gamma\,(B_1+B_2)$.

Corollary (1.8): Let M be an R- module . If $A\,\gamma\,B$ and C is a submodule of M , then $(A+C)\,\gamma\,(B+C)$. The converse is true when $C \ll M$

Proof: Assume that $A\,\gamma\,B$. Since $C\,\gamma\,C$, then by [1] , $(A+C)\,\gamma\,(B+C)$. Conversely, assume that $(A+C)\,\gamma\,(B+C)$, then $B+C \leq_{ce} A+B+C$ in M. Since $C \ll M$, then $B \leq_{ce} A+B$ in M, [1]. Thus $A\,\gamma\,B$.

Proposition (1.9): Let $f : M \longrightarrow N$ be an R-epimorphism and let A , B submodules of M. If $A\,\gamma\,B$, then $f(A)\,\gamma\,f(B)$.

Proof: Assume that $A\,\gamma\,B$, then $B \leq_{ce} A+B$ in M. Hence $f(B) \leq_{ce} f(A+B) = f(A)+f(B)$ in N , by [1]. Thus $f(A)\,\gamma\,f(B)$.

Proposition (1.10): Let $f : M \longrightarrow N$ be an R-epimorphism and let A , B submodules of M. If $A\,\gamma\,B$, then $f^{-1}(A)\,\gamma\,f^{-1}(B)$.

Proof: Suppose that $A\,\gamma\,B$ and let X be a submodule of M such that $M = f^{-1}(A) + X$. Then $N = A + f(X)$. But $A\,\gamma\,B$,therefore $N = B + f(X)$ and hence $M = f^{-1}(B)+X$. Thus $f^{-1}(A)\,\gamma\,f^{-1}(B)$.

Proposition (1.11): Let $M = M_1 \oplus M_2$ and let A and B be submodules of M_1 and M_2 respectively. Then $A\,\gamma\,M_1$ and $B\,\gamma\,M_2$ if and only if $(A \oplus B)\,\gamma\,(M_1 \oplus M_2)$.

Proof: ($\Rightarrow$) By prop. (1.7).

($\Leftarrow$) Let $P_1 : M \longrightarrow M_1$ and $P_2 : M \longrightarrow M_2$ be the projection homomorphisms on M_1 and M_2 respectively .Since $(A \oplus B)\,\gamma\,(M_1 \oplus M_2)$, then $A = P_1(A \oplus B)\,\gamma\,P_1(M_1 \oplus M_2) = M_1$ and $B = P_2(A \oplus B)\,\gamma\,P_2(M_1 \oplus M_2) = M_2$, by Prop. (1.9). Hence $A\,\gamma\,M_1$ and $B\,\gamma\,M_2$.

§2: Goldie lifting modules

In this section , we define a Goldie lifting modules with examples and basic properties.

Definition (2.1) Let$M$ be an R- module. We say that M is a Goldie lifting module (briefly $\mathcal{G}$-lifting) if for every proper submodule$A$ of M, there exists a proper direct summand D of M such that $A\,\gamma\,D$.

Remark (2.2): Every lifting module is $\mathcal{G}$-lifting.
Proof: Let M be a lifting module and let A be a proper submodule of M , there exists a direct summand D of M such that $D\leq A$ and$\frac{A}{D}\ll\frac{M}{D}$. Hence $D\leq_{ce} A$ in M. Since $A\neq M$, then $D\neq M$. By (1.2-2) ,$A\,\gamma\,D$. Thus M is$\mathcal{G}$-lifting.

The following example show that a $\mathcal{G}$-lifting module need not be a lifting module.

Example (2.3):
Consider the Z-module $M = Z_8 \oplus Z_2$. The proper submodules of M are:
$A_1 = \{(\bar{1},\bar{0}), (\bar{2},\bar{0}), (\bar{3},\bar{0}), (\bar{4},\bar{0}), (\bar{5},\bar{0}), (\bar{6},\bar{0}), (\bar{7},\bar{0}), (\bar{0},\bar{0})\}$.
$A_2 = \{(\bar{2},\bar{0}), (\bar{4},\bar{0}), (\bar{6},\bar{0}), (\bar{0},\bar{0})\}$.
$A_3 = \{(\bar{4},\bar{0}), (\bar{0},\bar{0})\}$.
$A_4 = \{(\bar{0},\bar{1}), (\bar{0},\bar{0})\}$.
$A_5 = \{(\bar{1},\bar{1}), (\bar{2},\bar{0}), (\bar{3},\bar{1}), (\bar{4},\bar{0}), (\bar{5},\bar{1}), (\bar{6},\bar{0}), (\bar{7},\bar{1}), (\bar{0},\bar{0})\}$.
$A_6 = \{(\bar{2},\bar{1}), (\bar{4},\bar{0}), (\bar{6},\bar{1}), (\bar{0},\bar{0})\}$.
$A_7 = \{(\bar{4},\bar{1}), (\bar{0},\bar{0})\}$.
$A_8 = \{(\bar{2},\bar{0}), (\bar{4},\bar{0}), (\bar{6},\bar{0}), (\bar{2},\bar{1}), (\bar{4},\bar{1}), (\bar{6},\bar{1}), (\bar{0},\bar{1}), (\bar{0},\bar{0})\}$.
$A_9 = \{(\bar{4},\bar{0}), (\bar{4},\bar{1}), (\bar{0},\bar{1}), (\bar{0},\bar{0})\}$.
$A_{10} = \{(\bar{0},\bar{0})\}$.
Clearly that $M= A_1 \oplus A_4= A_1 \oplus A_7= A_4 \oplus A_5$ and the small submodules of M areA_2 and A_3.
It is enough to check that A_6, A_8 and A_9 satisfy the definition. For A_6 , the only submodule$A$ of M satisfy $A_6 + A = M$ is A_1. Since A_1 is a direct summand of M , then $A_6\,\gamma\,A_4$ and $A_6\,\gamma\,A_7$.
For A_8 , since $A_1$and A_5 are satisfy $M = A_8 + A_1 = A_8 + A_5$ and both is a direct summand , then $A_8\,\gamma\,A_4$. By the same argument one can see that $A_9\,\gamma\,A_4$. Thus M is $\mathcal{G}$-lifting.
Claim that M is not lifting. To see this , consider the submoduleA_6. The only direct summand of M contained in A_6 is $\{(\bar{0},\bar{0})\}$. If M is lifting , then $A=D\oplus S$, where D is a direct summand of M and $S\ll M$. Hence $A_6\ll M$ which is a contradiction .Thus M is not lifting.

Examples (2.4):
1- Z_4as Z- module is $\mathcal{G}$-lifting.
2- Z as Z- moduleis not$\mathcal{G}$-lifting.To show that , consider the submodule$A= 2Z$ of Z. If Z is $\mathcal{G}$-lifting , then there exists a proper direct summand D of Z such that $A\,\gamma\,D$. But Z is indecomposable , so $D=0$ and hence $2Z\ll Z$, by (1.2) which is a contradiction.
3- Let Q be the set of the rational numbers. It is easy to see that Q as Z- module is not$\mathcal{G}$-lifting**.**

Recall that a non-zero R- module M is called a hollow module if every proper submodule of M is a small submodule of M, see [6].

The following proposition gives a condition under which the lifting module and $\mathcal{G}$-lifting module are equivalent.

Proposition (2.5): Let*M* be an indecomposable module. Then the following statements are equivalent.
1- *M* is lifting.
2- *M* is $\mathcal{G}$-lifting.
3- *M* is hollow.
Proof: (1)$\Rightarrow$(2) Remark (2.2).
(2)$\Rightarrow$(3) Suppose that *M* is $\mathcal{G}$-lifting module and let *A* be a proper submodule of *M*. Since *M* is $\mathcal{G}$-lifting , then there exists a proper direct summand *D* of *M* such that $A\,\gamma\,D$. But *M* is indecomposable , therefore $D=0$, then $A \ll M$, by remark (1.2). Thus *M* is hollow.
(3)$\Rightarrow$(1) Clear.

Proposition (2.6) Let*A* be a submodule of a$\mathcal{G}$-lifting module *M*. If $\frac{A+D}{A}$ is a direct summand of $\frac{M}{A}$, for every direct summand *D* of *M* , then $\frac{M}{A}$ is $\mathcal{G}$-lifting.
Proof: Assume that *M* is $\mathcal{G}$-lifting and let $\frac{B}{A}$ be a proper submodule of $\frac{M}{A}$. Note that $B\neq M$, if $B=M$, then $\frac{B}{A} = \frac{M}{A}$ which is a contradiction. Since *M* is $\mathcal{G}$-lifting , then there exists a proper direct summand *D* of *M* such that $B\,\gamma\,D$. By our assumption ,$\frac{A+D}{A}$ is a direct summand of $\frac{M}{A}$. Note that $\frac{A+D}{A} \neq \frac{M}{A}$, if $\frac{A+D}{A} = \frac{M}{A}$, then $M = A+D = B+D$. But $B\,\gamma\,D$, therefore $M=D$ which is a contradiction. To show $\frac{B}{A}\gamma\frac{A+D}{A}$. Let $\pi : M \longrightarrow \frac{M}{A}$ be the natural epimorphisim. Since $B\,\gamma\,D$, then $\pi(B)\ \gamma\ \pi(D)$, by Prop. (1.9). Hence $\frac{B}{A}\gamma\frac{A+D}{A}$. Thus $\frac{M}{A}$ is $\mathcal{G}$-lifting.
Let *M* be an *R*-module and let $A \leq M$. Recall that *A* is fully invariant submodule if $f(A) \leq A$,$\forall f \in End(M)$, see [7].

Proposition (2.7): Let*M* be a $\mathcal{G}$-lifting module , then $\frac{M}{A}\mathcal{G}$-lifting , for every fully invariant submodule*A* of *M*.
Proof: Let *A* be a fully invariant submodule of *M* and let $\frac{B}{A}$ be a proper submodule of $\frac{M}{A}$. Clearly that *B* is a proper submodule of *M*. Since *M* is$\mathcal{G}$-lifting , then there exists a proper direct summand *D* of *M* such that $B\,\gamma\,D$. But *A* is fully invariant , therefore $\frac{A+D}{A}$ is a direct summand of $\frac{M}{A}$, by [8, Lemma 5-4]. It is easy to see that $\frac{A+D}{A} \neq \frac{M}{A}$. Let $\pi : M \longrightarrow \frac{M}{A}$ be the natural epimorphisim. Since $B\,\gamma\,D$, then $\pi(B)\ \gamma\ \pi(D)$, by Prop. (1.9).Hence $\frac{B}{A}\gamma\frac{A+D}{A}$. Thus $\frac{M}{A}$ is $\mathcal{G}$-lifting.
Let *M* be an *R*- module. Recall that *M* is said to be Distributive module if $A\cap(B+C) = (A\cap B)+(A\cap C)$, for all submodules *A,B,C* of *M* , see [9].

Proposition (2.8) : Let *M* be distributive $\mathcal{G}$-lifting *R*- module and let *A* be a submodule of *M*. Then $\frac{M}{A}$is $\mathcal{G}$-lifting.
Proof: Let $\frac{B}{A}$ be a proper submodule of$\frac{M}{A}$. Since *M* is $\mathcal{G}$-lifting and *B* is proper submodule of *M* , then there exists a proper direct summand *D* of *M* such that $B\,\gamma\,D$. Let $M = D \oplus D'$, for some submoduleD' of *M*. Then $\frac{M}{A} = (\frac{A+D}{A}) + (\frac{D'+A}{A})$ and $(\frac{A+D}{A}) \cap (\frac{D'+A}{A}) = \frac{(D\cap D')+(D\cap A)+(A\cap D')+A}{A} = \frac{0+A\cap(D+D')+A}{A} = \frac{A}{A}$, because *M* is distributive. Hence $\frac{D+A}{A}$is a proper direct summand of $\frac{M}{A}$. One can easily show that $\frac{B}{A}\gamma\frac{A+D}{A}$. Thus $\frac{M}{A}\mathcal{G}$-lifting
Now , we give a various characterization of $\mathcal{G}$-lifting module.

Proposition(2.9): Let *M* be an *R*- module. Then the following statements are equivalent.
1- *M* is $\mathcal{G}$-lifting.

2- For every proper submodule A of M , there exists a proper direct summand D of M such that $M = D \oplus D'$ and $(A+D)\cap D' \ll D'$.
3- For every proper submodule A of M , there exists a proper direct summand D of M such that $A+D = D \oplus S$, $S \ll M$.

Proof: (1) $\Rightarrow$(2) Let A be a proper submodule of M. Since M is $\mathcal{G}$-lifting , then there exists a proper direct summand D of M such that $A\,\gamma\, D$. Let $M = D \oplus D'$,$D' \leq M$. To show $(A+D)\cap D' \ll D'$, let U be a submodule of D' such that $[(A+D)\cap D'] + U = D'$. So $M = D+D' = D + [(A+D)\cap D'] + U$. Now ,$\frac{M}{D} = \frac{D+U}{D} + \frac{[(A+D)\cap D']+D}{D}$. But $\frac{[(A+D)\cap D']+D}{D}$ is a submodule of $\frac{A+D}{D}$ and $\frac{A+D}{D} \ll \frac{M}{D}$, therefore $\frac{[(A+D)\cap D']+D}{D} \ll \frac{M}{D}$, by [3, Lemma 4.2 , P.56]. Hence $M = D+U$. Since $D\cap U \leq D\cap D' = 0$,then $D\cap U=0$. Hence $M = D \oplus U$. So $U=D'$ .Thus $(A+D)\cap D' \ll D'$.

(2) $\Rightarrow$(3) Let A be a proper submodule of M. By our assumption ,there exists a proper direct summand D of M such that $M = D \oplus D'$, $D' \leq M$ and $(A+D)\cap D' \ll D'$.Now ,$A+D = (A+D)\cap M = (A+D)\cap(D+D') = D \oplus [(A+D)\cap D']$, where D is a direct summand of M and $(A+D)\cap D' \ll D'$.

(3) $\Rightarrow$(1) Let A be a proper submodule of M. By our assumption ,there exists a proper direct summand D of M such that $A+D = D \oplus S$, $S \ll M$. Claim that $\frac{A+D}{D} \ll \frac{M}{D}$. To see this , let $\frac{U}{D} \leq \frac{M}{D}$ such that $\frac{M}{D} = \frac{A+D}{D} + \frac{U}{D}$, then $M = A+D+U = D+S+U = D+U = U$, hence $\frac{A+D}{D} \ll \frac{M}{D}$. Thus M is $\mathcal{G}$-lifting.

Proposition (2.10): Let M be an R- module. Then M is $\mathcal{G}$-lifting if and only if for each proper submodule A of M , there exists a proper direct summand D of M and a submodule B of M such that $A \leq B$ and $D \leq_{ce} B$ in M.

Proof: Suppose that M is a $\mathcal{G}$-lifting and let A be a proper submodule of M. Then there exists a proper direct summand D of M such that $D \leq_{ce} A+D$ in M. Let $B=A+D$. Thus we get the result.

Conversely, let A be a proper submodule of M. By our assumption , there exists a proper direct summand D of M and a submodule B of M such that $A \leq B$ and $D \leq_{ce} B$ in M. Since $D \leq A+D \leq B$ and $D \leq_{ce} B$ in M, then $D \leq_{ce} A+D$ in M. Thus M is $\mathcal{G}$-lifting.

Let M be an R- module. Recall that M is called a supplemented module, if every submodule of M has a supplement in M, see [6].

Proposition (2.11): Let M be an amply supplemented module. Then the following statements are equivalent:

1- M is $\mathcal{G}$-lifting module.
2- For each coclosed submodule A of M , there exists a proper direct summand D of M such that $A\,\gamma\, D$.

Proof: (1)$\Rightarrow$(2) Clear.

(2) $\Rightarrow$ (1) Let A be a proper submodule of M. Since M is amply supplemented , A has a coclosure submodule say B , by Prop.(3.1.9). Since B is coclosed submodule , there exists a proper direct summand D of M such that $B\,\gamma\, D$, by (2). Now since $D \leq_{ce} B+D \leq_{ce} A+D$ in M , then $D \leq_{ce} A+D$ in M , hence $A\,\gamma\, D$. Thus M is $\mathcal{G}$-lifting.

Note:

A direct sum of $\mathcal{G}$-lifting modules may not be $\mathcal{G}$-lifting. Now , we give sufficient conditions under which the direct sum of $\mathcal{G}$-lifting modules is $\mathcal{G}$-lifting.

Proposition(2.12): Let $M= M_1 \oplus M_2$ be an R- module such that $ann(M_1)+ ann(M_2)= R$. If M_1 and M_2 are $\mathcal{G}$-lifting modules.Then M is $\mathcal{G}$-lifting.

Proof: Let A be a proper submodule of M. By the same argument of the proof of [7 , prop. 4.2 , CH.1], $A=A_1 \oplus A_2$, where A_1 is a submodule of M_1 and A_2 is a submodule of M_2. Consider the case when A_1 and A_2 are proper submodules of M_1 and M_2 respectively. Since M_1 and M_2 are $\mathcal{G}$-lifting , then there exists a proper direct summands D_1 of M_1 and D_2 of M_2 such that $A_1 \gamma D_1$ and $A_2 \gamma D_2$. It is easy to show that $D_1 \oplus D_2$ is a proper direct summand of M , then by prop. (1.11) , $(A_1 \oplus A_2) \gamma (D_1 \oplus D_2)$. Now , if $A_1 = M_1$, then $A_2 \neq M_2$, there is a proper direct summand D_3 of M_2 such that $A_2 \gamma D_3$. Hence $(A_1 \oplus A_2) \gamma (M_1 \oplus D_3)$, where $(M_1 \oplus D_3)$ is a proper direct summand of M. By the same argument we can get the result when $A_2 = M_2$. Thus M is $\mathcal{G}$-lifting.

Proposition (2.13): Let $M=M_1 \oplus M_2$ be a duo module such that M_1 and M_2 are $\mathcal{G}$-lifting. modules. Then M is $\mathcal{G}$-lifting.

Proof: Assume that $M = M_1 \oplus M_2$ is a duo module and let A be a proper submodule of M, then ,by our assumption , A is fully invariant, hence $A=(A \cap M_1) \oplus (A \cap M_2)$. Consider the case when $(A \cap M_1)$ and $(A \cap M_2)$ are proper submodules of M_1 and M_2 respectively. Since M_1 and M_2 are $\mathcal{G}$-lifting , then there exists a proper direct summands D_1 of M_1 and D_2 of M_2 such that $(A \cap M_1) \gamma D_1$ and $(A \cap M_2) \gamma D_2$,then $(A \cap M_1) \oplus (A \cap M_2) \gamma (D_1 \oplus D_2)$, by prop. (1.11) , where $D_1 \oplus D_2$ is a proper direct summand of M , Now if $(A \cap M_1) = M_1$, then $(A \cap M_2) \neq M_2$, there is a proper direct summand D_3 of M_2 such that $(A \cap M_2) \gamma D_3$. Hence $(A \cap M_1) \oplus (A \cap M_2) \gamma (M_1 \oplus D_3)$, where $(M_1 \oplus D_3)$ is a proper direct summand of M. Similarly , we can get the result when $(A \cap M_2) = M_2$. Thus M is $\mathcal{G}$-lifting.

By the same argument one can prove the following proposition.

Proposition (2.14): Let $M= M_1 \oplus M_2$ be a distributive module such that M_1 and M_2 are $\mathcal{G}$-lifting modules. Then M is $\mathcal{G}$-lifting.

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