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On SAS-Injective Rings

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Abstract
Let R be a ring. A right R-module M is called SAS-N-injective (where N is any right R-module) if every right R-homomorphism from a semiartinian small right submodule of N into M extends to N. A ring R is called right SAS-injective if R_R is SAS-R-injective module. Right SAS-injective rings are studied in this paper. Many characterizations and properties of this type of rings are obtained.

Keywords: SAS-injective ring, finitely generated module, injective ring, semiartinian module, small submodule.

حول الحلقات الاغمارية من النمط- SAS

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الخلاصة
لتكن R حلقة. المقاس الايمن M على الحلقة R يسمى أغماري من النمط–SAS نسبة الى N (حيث N هو مقاس ايمن على الحلقة R) اذا كل تماثل مقاسي ايمن على الحلقة R من مقاس جزئي ايمن شبه ارتيني صغير من N الى M يوسع الى N. الحلقة R تسمى حلقة اغمارية يمنى من النمط- SAS اذا كان المقاس الايمن R_R هو مقاس اغماري من النمط–SAS نسبة الى الحلقة R. الحلقات الاغمارية اليمنى من النمط- SAS قد درست في هذا البحث. تم الحصول على العديد من تشخيصات وخصائص هذا النوع من الحلقات.

1. Introduction

Throughout this paper, R is an associative ring with identity 1 and any module is unitary. By a module (resp., homomorphism) we mean a right R-module (resp. right R-homomorphism), if not otherwise specified. The class of right R-modules is denoted by Mod-R. We write $J(M)$ and soc(M) for the Jacobson radical and the socle of a right R-module M, respectively. We write $Z(R_R)$ for the right singular ideal of a ring R. A module M is called semiartinian, if soc$(M/K) \neq 0$, for any proper submodule K of M [1]. For a right R-module M_R, we use $Sa(M)$ to denote the sum of all semiartinian submodules of M. A proper submodule A of a module M is called small, if $A + B = M$ where B is a submodule of M implies $B = M$ [1]. For any $a \in R$, we use $l_R(a)$ (resp. $r_R(a)$) to denote the left (resp., right) annihilator of a in R.

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Injective modules play important role in module theory, and extensively many authors are studied their generalizations (see, for example, [2-6]). If every R-homomorphism from a right ideal of a ring R into R_R can be extended to R_R, then R is called right self-injective ring [7]. Let $N, M \in$ Mod-R, then M is called an SAS-N-injective, if any right R-homomorphism from a semiartinian small right submodule of N into M extends to N. If a module M is an SAS-R-injective, then M is called an SAS-injective. A ring R is called a right SAS-injective if the right R-module R_R is an SAS-injective [8]. The SAS-injective rings have been studied in this paper. Many characterizations and properties of right SAS-injective rings have been obtained. For examples, we prove that a ring R is a right SAS-injective if and only if for any $N \in$ Mod-R and a non-zero R-monomorphsim f from N to R with $f(N)$ a semiartinian small right ideal of R, then $Hom_R(R, N) = Rf$. Also, we prove that if R is a right SAS-injective ring, then $l_R(A_1 \cap A_2) = l_R(A_1) + l_R(A_2)$, for every semiartinian small right ideals A_1 and A_2. Moreover, we show that if Rb is a minimal left ideal of a right SAS-injective ring R, then $J(bR) \cap \text{soc}(bR)$ is zero or simple, for any $b \in R$. Condition under which SAS-injectivity implies injectivity is given. We get that if R is a semiperfect ring, then R is a right SAS-injective ring if and only if any R-homomorphism from a semiartinian right ideal of R into R extends to R. We prove that if R is a right SAS-injective ring, and $a, b \in R$ with $b \in Sa(R_R) \cap J(R_R)$ and $aR \cong bR$, then $Ra \cong Rb$. Finally, we show that if R is a right SAS-injective ring, then the set $\{a \in R \,|\, r_R(1 - sa) = 0 \text{ for all } s \in R\}$ is contained in $D(R_R)$, where $D(R_R) = \{a \in R \,|\, r_R(a) \cap mR \neq 0 \text{ for each } 0 \neq m \in Sa(R_R) \cap J(R_R)\}$.

2. SAS-Injective Rings

Let $N, M \in$ Mod-R. Then M is called SAS-N-injective, if any right R-homomorphism from a semiartinian small right submodule of N into M extends to N. A right R-module M is called SAS-injective if M is SAS-R-injective. A ring R is called right SAS-injective if the right R-module R_R is SAS-injective [8]. In this section, right SAS-injective rings are studied extensively. Many characterizations and properties of this type of rings are given.

If any submodule N of a module M takes the form MI, for some ideal I of R, then M is called multiplication module [9].

A right R-module M is called projective if for any right R-epimorphism $f: A \to B$ and for any right R-homomorphism $h: M \to B$, there is a $g \in Hom_R(M, A)$ such that $hg = f$ [1, p. 117].

We begin this section with the following theorem, which gives some characterizations of right SAS-injective rings.

Theorem 2.1. Consider the following statements for a ring R:
(1)R is a right SAS-injective ring.
(2)If P and D are finitely generated projective right R-modules with K is a semiartinian small submodule of P, then any R-homomorphism $f: K \longrightarrow D$ can be extended to an R-homomorphism $g: P \longrightarrow D$.
(3)If $N \in$ Mod-R and f is a nonzero R-monomorphsim from N to R with $f(N)$ is a semiartinian small right ideal of R, then $Hom_R(N, R) = Rf$.
Then (2) $\Longrightarrow$ (1) and (1) $\Longleftrightarrow$ (3). Moreover, if a module R_R^m is multiplication for any $m \in \mathbb{Z}^+$, then (1) $\Longrightarrow$ (2).

Proof. (2) ⇒ (1) Clear.
(1) ⇒ (2) Let R be a right SAS-injective ring with R_R^m a multiplication module, for every $m \in \mathbb{Z}^+$. Let P and D be finitely generated projective modules and K a semiartinian small submodule of P. Let $f: K \longrightarrow D$ be any R-homomorphism. Since D is finitely generated, there exists a right R-epimorphism $\alpha_1: R^n \longrightarrow D$ for some $n \in \mathbb{Z}^+$. Projectivity of D implies that there is a right R-homomorphism $\alpha_2: D \longrightarrow R^n$ with $\alpha_1\alpha_2 = I_D$, where $I_D: D \longrightarrow D$ is the identity homomorphism. Thus from right SAS-injectivity of ring R and [8] we get that R^n is a right SAS-R^m-injective R-module, for any $m \in \mathbb{Z}^+$. Since P is finitely generated projective, P is a direct summand of R^k, for some $k \in \mathbb{Z}^+$. By [8], R^n is SAS-P-injective. Then $hi = \alpha_2 f$, for some $h \in Hom_R(P, R^n)$. Put $g = \alpha_1 h: P \longrightarrow D$. Then $gi = (\alpha_1 h)i = \alpha_1(hi) = \alpha_1(\alpha_2 f) = (\alpha_1\alpha_2)f = I_D f = f$. Therefore, $gi = f$ for some R-homomorphism $g: P \longrightarrow D$.
(1) ⇒ (3) Let R be a right SAS-injective ring. Let N be any right R-module and $f: N \longrightarrow R$ be a nonzero R-monomorphism with $f(N)$ is a semiartinian small right ideal of R. Define $\acute{f}: N \longrightarrow f(N)$ by $\acute{f}(x) = f(x)$, for any $x \in N$. It is clear that $\acute{f}$ is an isomorphism. Let $g \in Hom_R(N, R)$, then we get $g\acute{f}^{-1}: f(N) \longrightarrow R$ is an R-homomorphism. Since R is a right SAS-injective ring (by hypothesis) and $f(N)$ is a semiartinian small right ideal of R, there is $c \in R$ with $(g\acute{f}^{-1})(k) = ck$, for all $k \in f(N)$ (by [8, Proposition 2.7]). Let $x \in N$, then $f(x) \in f(N)$ and hence $(g\acute{f}^{-1})(f(x)) = cf(x)$. Since $(g\acute{f}^{-1})(f(x)) = g(x)$, it follows that $g(x) = cf(x)$, for any $x \in N$. Thus $Hom_R(N, R) = Rf$.
(3) ⇒ (1) Let K be a semiartinian small right ideal of R, $f: K \longrightarrow R$ a right R-homomorphism, and $i: K \longrightarrow R$ the inclusion map. Then by hypothesis, we have $Hom_R(K, R) = Ri$ and hence $f = ci$ for some $c \in R$. Thus there exists $c \in R$ such that $f(a) = ca$ for all $a \in K$. Then R is a right SAS-injective ring, by [8]. □

Theorem 2.2. Let R be a right SAS-injective ring. Then the following statements hold:
(1) $l_R r_R(m) = Rm$, for all $m \in Sa(R_R) \cap J(R_R)$.
(2) If $r_R(m) \subseteq r_R(n)$, where $m \in Sa(R_R) \cap J(R_R)$ and $n \in R$, then $Rn \subseteq Rm$.
(3) $l_R\big(mR \cap r_R(a)\big) = l_R(m) + Ra$, for all $m, a \in R$ with $am \in Sa(R_R) \cap J(R_R)$.
(4) If $f: aR \longrightarrow R$, $a \in Sa(R_R) \cap J(R_R)$, is a right R-homomorphism, then $f(a) \in Ra$.

Proof. (1) Let $m \in Sa(R_R) \cap J(R_R)$ and $\in l_R r_R(m)$. By [10, Proposition 2.15, p. 37], $r_R(m) = r_R l_R r_R(m) \subseteq r_R(n)$. Let $f: mR \longrightarrow R$ is given by $f(mr) = nr$ for any $r \in R$. Then f is a well-defined right R-homomorphism. By hypothesis, there is an endomorphism g of R with $g(x) = f(x)$, for any $x \in mR$. Therefore, $n = n.1 = f(m.1) = f(m) = g(m) = g(1)m \in Rm$. Hence $l_R r_R(m) \subseteq Rm$. Conversely, let $rm \in Rm$, where $r \in R$. Thus $rmk = 0$ for all $k \in r_R(m)$ and hence $rm \in l_R r_R(m)$. Therefore, $l_R r_R(m) = Rm$.
(2) Let $n \in R$ and $m \in Sa(R_R) \cap J(R_R)$ such that $r_R(m) \subseteq r_R(n)$. Thus $n \in l_R r_R(n)$. Since $r_R(m) \subseteq r_R(n)$ (by hypothesis), $l_R r_R(n) \subseteq l_R r_R(m)$ (by [10, Proposition 2.15, p. 37]). So, $\in l_R r_R(m)$. By (1), $n \in Rm$ and this implies that $Rn \subseteq Rm$.
(3) Let $a, m \in R$ such that $am \in Sa(R_R) \cap J(R_R)$. If $x \in l_R(m) + Ra$, then $x = x_1 + x_2$ such that $x_1 m = 0$ and $x_2 = sa$ for some $s \in R$. For all $b \in mR \cap r_R(a)$, we have $b = mr$ and $ab = 0$ for some $r \in R$. Since $x_1 b = x_1(mr) = (x_1 m)r = 0$ and $x_2 b = (sa)b = s(ab) = 0$, it follows that $x \in l_R\big(mR \cap r_R(a)\big)$ and this implies that $l_R(m) + Ra \subseteq l_R\big(mR \cap r_R(a)\big)$. Let $y \in l_R\big(mR \cap r_R(a)\big)$. If $r \in r_R(am)$, then $(am)r = 0$ and hence $a(mr) = 0$. Thus $mr \in mR \cap r_R(a)$ and hence $(ym)r = y(mr) = 0$ and so $ym \in l_R(r_R(am))$. Thus $r_R l_R(r_R(am)) \subseteq r_R(ym)$. By [10, Proposition 2.15, p. 37], $r_R(am) \subseteq r_R(ym)$. By hypothesis, $am \in Sa(R_R) \cap J(R_R)$. By (2), $Rym \subseteq Ram$. Thus $ym = sam$, for some $s \in R$

and hence $(y - sa)m = 0$ and this implies that $y - sa \in l_R(m)$. Thus $y \in l_R(m) + Ra$ and hence $l_R(mR \cap r_R(a)) = l_R(m) + Ra$.

(4) Let $a \in Sa(R_R) \cap J(R_R)$ and let $f: aR \longrightarrow R$ be a right R-homomorphism. Put $d = f(a)$, then $r_R(a) \subseteq r_R(d)$. By (2), $Rd \subseteq Ra$ and hence $f(a) \in Ra$. □

Proposition 2.3. If R is a right SAS-injective ring, then $l_R(A_1 \cap A_2) = l_R(A_1) + l_R(A_2)$, for any semiartinian small right ideals A_1 and A_2 of R.

Proof. Let A_1 and A_2 be any two semiartinian small right ideals of R. Let $r \in l_R(A_1 \cap A_2)$, thus $r.(A_1 \cap A_2) = 0$. Consider the mapping $f: A_1 + A_2 \longrightarrow R$ is given by $f(a_1 + a_2) = r.a_1$, for all $a_1 \in A_1$, $a_2 \in A_2$. Thus f is a well-defined right R-homomorphism, since if $a_1 + a_2 = b_1 + b_2$, where $a_1, b_1 \in A_1$, $a_2, b_2 \in A_2$, then $a_1 - b_1 = b_2 - a_2 \in A_1 \cap A_2$. Since $r(A_1 \cap A_2) = 0$, we have that $r(a_1 - b_1) = 0$ and hence $ra_1 = rb_1$, so $f(a_1 + a_2) = f(b_1 + b_2)$ and this implies that f is a well-defined. Also, for every $a_1 + a_2$, $b_1 + b_2 \in A_1 + A_2$, where $a_1, b_1 \in A_1$, $a_2, b_2 \in A_2$ and $t \in R$, we have $f\big((a_1 + a_2) + (b_1 + b_2)\big) = f\big((a_1 + b_1) + (a_2 + b_2)\big) = r(a_1 + b_1) = ra_1 + rb_1 = f\big((a_1 + b_1) + (a_2 + b_2)\big)$ and $f((a_1 + a_2)t) = f(a_1 t + a_2 t) = r(a_1 t) = (ra_1)t = (f(a_1 + a_2))t$. Thus, f is a well-defined right R-homomorphism. Since $A_1 + A_2$ is a semiartinian small right ideal of R_R, we get from SAS-injectivity of R_R that there is a right R-homomorphism $g: R \longrightarrow R$ such that $g(a_1 + a_2) = f(a_1 + a_2)$, for all $a_1 \in A_1$, $a_2 \in A_2$. Thus $g(a_1 + a_2) = ra_1$, so $ra_1 - g(a_1) = g(a_2) = g(0 + a_2) = r.0 = 0$ and hence $\big(r - g(1)\big)a_1 = 0$, for all $a_1 \in A_1$. So $r - g(1) \in l_R(A_1)$. Since $g(1) \in l_R(A_2)$ (because $g(1)A_2 = g(A_2) = 0$), we have that $r \in l_R(A_1) + l_R(A_2)$ and hence $l_R(A_1 \cap A_2) \subseteq l_R(A_1) + l_R(A_2)$. From [10, Proposition 2.16, p. 38], the other inclusion is obtained . □

A submodule N of a right R-module M is called essential in M, denoted by $N \subseteq^{ess} M$, if for every submodule K of M with $N \cap K = 0$, then $K = 0$ [1, p. 106].

Proposition 2.4. If R is a right SAS-injective ring, then $Sa(R_R) \cap J(R_R) \subseteq Z(R_R)$.

Proof. Let $a \in Sa(R_R) \cap J(R_R)$ and $mR \cap r_R(a) = 0$ for any $m \in R$. Thus from Theorem 2.2(3), we have that $l_R(m) + Ra = l_R\big(mR \cap r_R(a)\big) = l_R(0) = R$. Since $a \in Sa(R_R) \cap J(R_R)$, we have from [1, Corollary 9.1.3, p.214] that $l_R(m) = R$ and hence $m = 0$. So, $r_R(a) \subseteq^{ess} R$ and hence $a \in Z(R_R)$. Therefore, $Sa(R_R) \cap J(R_R) \subseteq Z(R_R)$. □

A ring R is called reduced if R has no non-zero nilpotent elements [7, p.249], where an element $a \in R$ is called nilpotent if $a^n = 0$ for some $n \in \mathbb{Z}^+$.

Corollary 2.5. If R is an SAS-injective reduced ring, then every right R-module is SAS-injective.

Proof. Let R be an SAS-injective reduced ring. By [7, Lemma 7.8, p. 249], $Z(R_R) = 0$. By Proposition 2.4, $Sa(R_R) \cap J(R_R) \subseteq Z(R_R)$ and hence $Sa(R_R) \cap J(R_R) = 0$. By [8], every right R-module is an SAS-injective. □

If for each sequence $a_1, a_2, a_3, \ldots$ of elements of a subset K of a ring R, we have $a_n \ldots a_2 a_1 = 0$, for some $n \in \mathbb{N}$, then K is called a right t-nilpotent [11, p.239].

Proposition 2.6. Let R be a right SAS-injective ring. If the ascending chain $r_R(a_1) \subseteq r_R(a_2 a_1) \subseteq \cdots \subseteq r_R(a_n \ldots a_2 a_1) \subseteq \cdots$ terminates for any sequence $a_1, a_2, \ldots$ in $Sa(R_R) \cap Z(R_R)$, then $Sa(R_R) \cap Z(R_R)$ is a right t-nilpotent and $Sa(R_R) \cap J(R_R) = Sa(R_R) \cap Z(R_R)$.

Proof. Let $a_1, a_2, \ldots$ be any sequence in $Sa(R_R) \cap Z(R_R)$, then we have $r_R(a_1) \subseteq r_R(a_2a_1) \subseteq \cdots$. By hypothesis, there exists $m \in \mathbb{N}$ such that $r_R(a_m \ldots a_2a_1) = r_R(a_{m+1}a_m \ldots a_2a_1)$. Assume that $a_m \ldots a_2a_1 \neq 0$. Since $r_R(a_{m+1}) \subseteq^{ess} R_R$, then $(a_m \ldots a_2a_1)R \cap r_R(a_{m+1}) \neq 0$ and hence $0 \neq a_m \ldots a_2a_1r \in r_R(a_{m+1})$ for some $r \in R$. Then $a_{m+1}a_m \ldots a_2a_1r = 0$ and this means that $a_m \ldots a_2a_1r = 0$ and this is a contradiction. Hence $Sa(R_R) \cap Z(R_R)$ is a right t-nilpotent and so $Sa(R_R) \cap Z(R_R) \subseteq J(R_R)$. Since $Sa(R_R) \cap J(R_R) \subseteq Z(R_R)$ (by Proposition 2.4), we have $Sa(R_R) \cap J(R_R) = Sa(R_R) \cap Z(R_R)$. □

Proposition 2.7. If R is a right SAS-injective ring, then $\mathrm{soc}(bR) \cap J(bR)$ is zero or simple for any $b \in R$ with Rb is a minimal left ideal of R.

Proof. Let $b \in R$ and suppose that $\mathrm{soc}(bR) \cap J(bR)$ is non-zero. Assume that $\mathrm{soc}(bR) \cap J(bR)$ is not simple. Thus there exist simple submodules x_1R and x_2R of $J(bR)$ with $x_i \in bR$ ($i = 1,2$). Thus $x_1R \cap x_2R = 0$. By Proposition 2.3, $l_R(x_1) + l_R(x_2) = R$. Since $x_i \in bR$, it follows that $x_i = br_i$ for some $r_i \in R$, $i = 1,2$ that is $l_R(b) \subseteq l_R(br_i) = l_R(x_i)$, $i = 1,2$. Since Rb is minimal, $l_R(b)$ is a maximal left ideal in R, that is $l_R(x_1) = l_R(x_2) = l_R(b)$ (because $l_R(x_i) \subsetneq R$). Therefore, $l_R(b) = R$ and hence $b = 0$ and this a contradiction with minimality of Rb. Hence $\mathrm{soc}(bR) \cap J(bR)$ is a simple right ideal of R.

Proposition 2.8. Let R be a right SAS-injective ring with $Sa(R_R) \cap J(R)$ is a semisimple right ideal of R. Then $r_Rl_R(Sa(R_R) \cap J(R_R)) = Sa(R_R) \cap J(R)$ if and only if $r_Rl_R(N) = N$ for every semiartinian small right ideal N of R.

Proof. (⇒) Assume that $r_Rl_R(Sa(R_R) \cap J(R_R)) = Sa(R_R) \cap J(R_R)$ and let N be a semiartinian small right ideal of R. We obtain $N \subseteq r_Rl_R(N)$ (by [10, Proposition 2.15 (2), p. 37]). We will prove that $N \subseteq^{ess} r_Rl_R(N)$. If $N \cap xR = 0$ for some $x \in r_Rl_R(N)$, then by Proposition 2.3, $l_R(N \cap xR) = l_R(N) + l_R(xR) = R$, since $x \in r_Rl_R(N) \subseteq r_Rl_R(Sa(R_R) \cap J(R_R)) = Sa(R_R) \cap J(R_R)$. If $y \in l_R(N)$, then $yx = 0$, and hence $y(xr) = 0$, for all $r \in R$ and so $l_R(N) \subseteq l_R(xR)$. Thus $l_R(xR) = R$ and hence $x = 0$ and this means $N \subseteq^{ess} r_Rl_R(N)$. Since $r_Rl_R(N) \subseteq r_Rl_R\big(Sa(R_R) \cap J(R)\big) = Sa(R_R) \cap J(R_R)$ and $Sa(R_R) \cap J(R_R)$ is semisimple (by hypothesis), we have that $r_Rl_R(N)$ is semisimple and hence $N = r_Rl_R(N)$.
(⇐) Suppose that $N = r_Rl_R(N)$ for all semiartinian small right ideal N of R. Since $Sa(R_R) \cap J(R_R)$ is a semiartinian small right ideal of R, $r_Rl_R\big(Sa(R_R) \cap J(R_R)\big) = Sa(R_R) \cap J(R_R)$ (by hypothesis). □

Remark 2.9. If R is a right SAS-injective ring, then it is not necessary that $J(R_R) \subseteq Sa(R_R)$, for example, let R be the localization ring of $\mathbb{Z}$ at the prime p, that is $R = \mathbb{Z}_{(p)} = \{\frac{m}{n}: \; p$ does not divide $n\}$. By [8], we have $Sa(R_R) = 0$ and R is a right SAS-injective ring, but $J(R_R) \neq 0$ and hence $J(R_R) \nsubseteq Sa(R_R)$.

If every R-homomorphism from a small right ideal of R into a module M can be extended to R_R, then M is called small injective [4].

Proposition 2.10. If a ring R is right SAS-injective with $J(R_R) \subseteq Sa(R_R)$, then R_R is a small injective module.

Proof. Let R be an SAS-injective ring with $J(R_R) \subseteq Sa(R_R)$. Thus $Sa(R_R) \cap J(R_R) = J(R_R)$. We will prove that R_R is a small-injective module. Let K be a small right ideal of R and $f: K \to R$ a right R-homomorphism. By [1, Theorem 9.1.1(a), p.213], $J(R_R)$ is the sum of all small right ideals in R and hence $K \subseteq J(R_R)$. Thus $K \subseteq Sa(R_R) \cap J(R_R)$ and hence K is a semiartinian small right ideal in R. By SAS-injectivity of R, the homomorphism f extends to R and hence R_R is a small-injective module. □

Let W be a right ideal of ring R. If $R_R = K_R \oplus L_R$ with K is a submodule of W and $L \cap W$ is a small right ideal of R, then W is called lies over a summand of R_R [12].

Proposition 2.11. If R is a right SAS-injective ring, then every R-homomorphism from W into R_R can be extended to an endomorphism of R_R, where W is a semiartinian right ideal lies over a summand of R_R.

Proof. Let $f: W \longrightarrow R$ be a right R-homomorphism. Since W lies over a summand of R_R (by hypothesis), it follows from [12] that there exists an idempotent $e^2 = e \in W$ such that $W = eR \oplus B$ for some right ideal $B \subseteq J(R)$. Since W is semiartinian, B is a semiartinian small right ideal of R. We need to prove that $W = eR \oplus (1-e)B$. Clearly, $eR + (1-e)B$ is direct sum, since if $x \in eR \cap (1-e)B$, then $x = er$ and $x = (1-e)b$, for some $b \in B$ and hence $b = er + eb \in eR \cap B = 0$. Thus $b = 0$ and hence $x = 0$, so $eR \cap (1-e)B = 0$. Let $x \in W$, then $x = a + b$, for some $a \in eR$, $b \in B$, we can write $x = a + eb + (1-e)b$ and hence $x \in eR \oplus (1-e)B$. Conversely, let $x \in eR \oplus (1-e)B$. Thus, $x = a + (1-e)b$, for some $a \in eR$ and $(1-e)b \in (1-e)B$, we get that $x = a + (1-e)b = a - eb + b \in eR \oplus B$. Hence $W = eR \oplus (1-e)B$. It is obvious that $(1-e)B$ is a semiartinian small right ideal of R. Let $f' : (1-e)B \longrightarrow R$ be a right R-homomorphism defined by $f'(x) = f(x)$, for all $x \in (1-e)B$. Since R is a right SAS-injective ring, there exists an R-homomorphism $g: R_R \longrightarrow R_R$ with $g\big((1-e)b\big) = f'((1-e)b)$ for all $(1-e)b \in (1-e)B$. Define $\alpha: R_R \longrightarrow R_R$ by $\alpha(x) = f(ex) + g((1-e)x)$, for each $x \in R$. Then α is a well-defined R-homomorphism. If $x \in W$, then $x = a + b$ where $a \in eR$ and $b \in (1-e)B$ and hence $\alpha(x) = f(ex) + g\big((1-e)x\big) = f(a) + g(b) = f(a) + f(b) = f(a+b) = f(x)$. □

A ring R is called semiperfect if $R/J(R)$ is semisimple and the idempotents of $R/J(R)$ can be lifted to R [7, p.363].

Corollary 2.12. If R is a semiperfect ring, then R is a right SAS-injective ring if and only if any R-homomorphism from a semiartinian right ideal of R into R extends to R.

Proof. Let R be a semiperfect ring. By [13], every right ideal lies over a summand of R.
($\Longrightarrow$) Let I be a semiartinian right ideal of a right SAS-injective ring R and $f: I \longrightarrow R$ a right R-homomorphism. By hypothesis, I lies over a summand of R. By Proposition 2.11, there is an R-homomorphism $g: R \longrightarrow R$ such that $g(a) = f(a)$, for all $a \in I$.
($\Longleftarrow$) It is clear. □

Theorem 2.13. Let R be a right SAS-injective ring, and let $a, b \in R$ with $b \in Sa(R_R) \cap J(R_R)$.

(1) If bR embeds in aR, then Rb is an image of Ra.
(2) If aR is an image of bR, then Ra embeds in Rb.
(3) If $bR \cong aR$, then $Ra \cong Rb$.

Proof. Assume that $f: bR \longrightarrow aR$ is a right R-homomorphism. Since $b \in Sa(R_R) \cap J(R_R)$ (by hypothesis), it follows from SAS-injectivity of R_R that there is an R-homomorphism $g: R \longrightarrow R$ such that $gi_1 = i_2 f$, where $i_1: bR \to R$ and $i_2: aR \to R$ are the inclusion maps. Thus $f(b) = g(b) = g(1)b = vb$, where $v = g(1)$. Since $f(b) \in aR$, it follows that $vb \in aR$ and

hence there is $u \in R$ such that $vb = au$. Define $\theta: Ra \longrightarrow Rb$ by $\theta(ra) = (ra)u = r(vb)$, for any $r \in R$. Thus θ is a well-defined left R-homomorphism.
(1) If f is a right monomorphism, we have $r_R(vb) \subseteq r_R(b)$. By Theorem 2.2.(2), $Rb \subseteq Rvb$. Thus $b = r(vb) = \theta(ra)$ (for some $r \in R$). Hence θ is a left R-epimorphism.
(2) If f is an epimorphism, then there is $s \in R$ such that $f(bs) = a$ and hence $a = f(b)s = vbs$. We will prove $\ker(\theta) = 0$. Let $x \in \ker(\theta)$, thus $\theta(x) = 0$. Since $x \in Ra$, we have $x = ra$, for some $r \in R$. Thus $\theta(ra) = 0$ and hence $r(vb) = 0$. So, $x = ra = r(bvs) = (rvb)s = 0$ and hence $\ker(\theta) = 0$. By [1, Lemma 3.1.8, p.44], θ is a left R-monomorphism.
(3) If f is an isomorphism, then by the proofs of (1) and (2), we have that θ is a left R-isomorphism. □

Lemma 2.14. Let R be ring, then $D(R_R) = \{a \in R \,|\, r_R(a) \cap mR \neq 0$ for each $0 \neq m \in Sa(R_R) \cap J(R_R)\}$ is a left ideal of R.

Proof. It is obviously that $D(R_R)$ is a non-empty set, since $0 \in D(R_R)$. If $a \in D(R_R)$ and $0 \neq m \in Sa(R_R) \cap J(R_R)$, thus $mb \in r_R(a) \cap mR$, for some $b \in R$ and so $a(mb) = 0$. Since $(-a)(mb) = -(amb) = 0$, then $mb \in r_R(-a)$ and hence $r_R(-a) \cap mR \neq 0$. Thus $-a \in D(R_R)$. Now, let $a_1, a_2 \in D(R_R)$ and $0 \neq m \in Sa(R_R) \cap J(R)$. We have that $0 \neq mb \in r_R(a_1) \cap mR$ for some $b \in R$. Since $a_2 \in D(R_R)$, it follows that $-a_2 \in D(R_R)$ and hence $0 \neq mbc \in r_R(-a_2) \cap mR$ for some $c \in R$. Therefore, $0 \neq mbc \in r_R(a_1) \cap r_R(-a_2) \cap mR$. Since $r_R(a_1) \cap r_R(-a_2) = r_R\big(a_1 + (-a_2)\big) = r_R(a_1 - a_2)$ (by [10, Proposition 2.16, p. 38]), we have $r_R(a_1 - a_2) \cap mR \neq 0$ for all $0 \neq m \in Sa(R_R) \cap J(R_R)$ and hence $a_1 - a_2 \in D(R_R)$. Also, let $x \in R$ and $a \in D(R_R)$. Since $r_R(a) \subseteq r_R(xa)$, it follows that $r_R(xa) \cap mR \neq 0$ for all $0 \neq m \in Sa(R_R) \cap J(R_R)$, that is $xa \in D(R_R)$. Thus $D(R_R)$ is a left ideal of R. □
Proposition 2.15. Let R be a right SAS-injective ring. Then $r_R(a) \subsetneq r_R(a - axa)$, for all $a \notin D(R_R)$ and for some $x \in R$.

Proof. For all $a \notin D(R_R)$, we can find $0 \neq m \in Sa(R_R) \cap J(R_R)$ such that $r_R(a) \cap mR = 0$. Clearly, $r_R(am) = r_R(m)$, so $Rm = Ram$ by Theorem 2.2(2). Thus $m = xam$ for some $x \in R$ and this implies that $m - xam = 0$ and hence $(1 - xa)(m) = 0$. Thus $a.(1 - xa)(m) = a.0$ and so $(a - axa)m = 0$. Therefore, $m \in r_R(a - axa)$, but $m \notin r_R(a)$ because $r_R(a) \cap mR = 0$ and hence the inclusion is strictly. □

Proposition 2.16. If R is a right SAS-injective ring, then the set $\{a \in R \,|\, r_R(1 - sa) = 0$ for all $s \in R\}$ is contained in $D(R_R)$.

Proof. We will prove that by contradiction. Assume that there is an element $a \in R$ such that $r_R(1 - sa) = 0$ for all $s \in R$ with $a \notin D(R_R)$. Then there exists $0 \neq m \in Sa(R_R) \cap J(R_R)$ with $r_R(a) \cap mR = 0$. If $r \in r_R(am)$, then $(am)r = 0$ and hence $a(mr) = 0$ and so $mr \in r_R(a)$. Since $r_R(a) \cap mR = 0$, it follows that $mr = 0$ and so $r \in r_R(m)$. Hence $r_R(am) \subseteq r_R(m)$. By Theorem 2.2(2), $Rm \subseteq Ram$. Thus $m = sam$, for some $s \in R$. Therefore, $(1 - sa)m = 0$ and hence $m \in r_R(1 - sa) = 0$ so $m = 0$ and this is a contradiction. Thus the statement is hold. □

3. Conclusions

The SAS-injective rings have been studied in this paper. Let $N, M \in$ Mod-R, then M is called an SAS-N-injective, if any right R-homomorphism from a semiartinian small right submodule of N into M extends to N. If a module M is an SAS-R-injective, then M is called an SAS-injective. A ring R is called a right SAS-injective if the right R-module R_R is an SAS-injective [8]. Many characterizations and properties of right SAS-injective rings have been obtained. For examples, we prove that a ring R is a right SAS-injective if and only if for any $N \in$ Mod-R and a non-zero R-monomorphsim f from N to R with $f(N)$ a semiartinian small right ideal of R, then $Hom_R(R, N) = Rf$. Also, we prove that if R is a right SAS-injective ring, then $l_R(A_1 \cap A_2) = l_R(A_1) + l_R(A_2)$, for every semiartinian small right ideals A_1 and A_2. Condition under which SAS-injectivity implies injectivity is given. We prove that if R is a semiperfect ring, then R is a right SAS-injective ring if and only if any R-homomorphism from a semiartinian right ideal of R into R extends to R. Finally, we show that if R is a right SAS-injective ring, then the set $\{a \in R \mid r_R(1 - sa) = 0$ for all $s \in R\}$ is contained in $D(R_R)$, where $D(R_R) = \{a \in R \mid r_R(a) \cap mR \neq 0$ for each $0 \neq m \in Sa(R_R) \cap J(R_R)\}$.

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