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Endo -Approximately 2-Absorbing Submodule and Module

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Abstract:
In this paper we introduce the concept of *Endo*-approximately 2-absorbing sub-module (shortly EATA) as a generalization of *Endo* 2-absorbing sub-module , where A proper sub-module N of an R-module M is called an EATA sub-module if for all $f, g \in \mathrm{End}M$, $x \in M$, $f \circ g(x) \in N$ implies the $f^2(x) \in N$ or $g^2(x) \in N$ or $f^2 \circ g^2(M) \subseteq N$. Many basic properties, characterization and examples of this concept are given.

Keywords: 2-absorbing sub-module, approximately 2-absorbing sub- module, prime sub -module, *Endo* 2-absorbing sub-module.

المقاسات والمقاسات الجزئية المستحوذة تقريبا على 2 من النمط Endo

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الخلاصة:
في هذا البحث قدمنا المقاسات الجزئية المستحوذة تقريبا على 2 من النمط Endoو المقاسات المستحوذة تقريبا على 2 من النمط Endo العديد من الخواص الأساسية والوصف والامثلة أعطيت لهذا المفهوم.

1. Introduction

Let M be an R-module over a commutative ring R. Prime sub- modules play vital part in module theory over combative ring. A prime sub-module N of M is a proper sub- module if $a \in R$, $x \in M$, and $ax \in N$, then $x \in N$ or $a \in (N:M)$[1]. A proper sub- module N of an R-module M is called S-prime (*Endo* prime), If for all $f \in S = \mathit{End}M$, $f(m) \in N$ implies that $m \in N$ or, $f(M) \subseteq N$ [2]. A S-prime module and quasi-Dedekind module are equivalent. An R-module M is called a quasi-Dedekind if there exists $f \in \mathit{End}M$, with $f \neq 0$, then $\ker f = 0$ [3]. An R-module M is called a scalar module if $\forall f \in \mathit{End}M$, there is $r \in R$ such that $f(m) = rm$, for all $m \in M$ [4]. We say that a module M over an integral domain R is called a torsion module (torsion free) if $T(M) = M$ ($T(M) = 0$, respectively) where $T(M) = \{m \in M : \exists s\ r \in R, r \neq 0, rm = 0\}$, [5]. Let R be an integral domain a module M is called divisible if, for every $0 \neq r \in R$ then $rM = M$ [6]. Also, An R- module M is said to be a multiplication module if for every sub-module N of M there is s an ideal I of R such that $N = IM$, [7]. Recall that M is said to be fully stable if every sub-module of M is stable, [8]. Suppose that R is a ring and M a right R-module. A sub-module N of M is called fully-invariant if f(N) is contained in N, [9].

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An R- module L is called M-injective module if any diagram with exact row: can be extended commutatively by a homomorphism

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[dl, dashed, "h"] \\ & L & \end{tikzcd}$$

$h: M \rightarrow L$ that is $h \circ f = g$, [10]. A sub-module N of R-module M is called an RD sub-module, if $rM \cap N = rN$, [11].

A sub-module N is a 2-absorbing sub- module of an R- module M if for any r, s∈ R, x∈ M with rsx ∈ N, and either rs ∈ (N: M) or, rx ∈ N or sx ∈ N [12]. Abdulrahman [13] studied of an *Endo* 2-absorbing sub- module. A proper sub-module N of an R-module M is said to be an *Endo* approximately 2-absorbing sub- module, if for all $f, g \in End$M, x∈ M, f∘g(x) ∈ N implies that f(x) ∈ N or g(x) ∈ N or f ∘ g (M) ⊆ N.

In this article we study the concept of *Endo*-approximately 2-absorbing sub-modules and module, as a generalization of *Endo* 2-absorbing sub-module, see Remark 2.2. Also, we give another Generalization of an Endo 2-absorbing sub-module, that every *Endo*- approximately 2-absorbing sub-module is approximately 2-absorbing sub-module, but the converse is not true (see Proposition 2.3 and Example 2.4). We clear some relationships between approximately 2-absorbing sub-modules and *Endo*- approximately 2-absorbing sub--modules, see Propositions 2.5.

2. EATA sub-module and module.

Definition2.1: A proper sub- module N of an R-module M is named an EATA sub- module if for all $f, g \in End$M, m∈ M, f ∘ g(m) ∈ N implies the f^2(m) ∈ N or g^2(m) ∈ N or, $f^2 \circ g^2$ (M) ⊆ N.

Remark 2.2:
(1) each *Endo* 2- absorbing sub- module is EATA sub-module. But the converse is not correct in general. For instance:
Let M be the Z - module $Z_6 \oplus Z_6$ and N ={(0,0)}.
Let f(x,y)= (0,x), g(x,y)= (2y,x). So, f∘g(3,3) = f(6,3)= f(0,3)= (0,0) ∈ N, and f^2(3,3) = f(0,3)= (0,0) ∈ N and g^2(3,3) = g(0,3)= (0,0) ∈ N, but f(3,3)= (0,3) ∉ N , g(3,3)= (0,3) ∉ N, and f∘g($Z_6 \oplus Z_6$) = f($2Z_6 \oplus Z_6$) =(0, $2Z_6$)=(0,2) ∉ N so{(0,0)} which is not *Endo* 2- absorbing sub- module.
(2) It is obviously that each *Endo*-prime sub-module is an EATA sub- module. We give later an example to show that the converse is not true in general.

Proposition 2.3: Let N be an *Endo* - approximately 2- absorbing sub- module of an R-module M, then N is approximately 2- absorbing sub- module of M.

Proof: Let abx ∈ N where a, b ∈R, x∈ M.
State f_1, f_2,:M → M such that f_1(x) =ax, f_2 (x)=bx for all x∈ M then abx = f_1 f_2 (x) ∈ N. Since N is *Endo* - approximately 2- absorbing, then either ${f_1}^2$(x) =a^2x ∈ N or, ${f_2}^2$(x)=b^2x∈ N

or, ${f_1}^2{f_2}^2(M) = a^2b^2M \subseteq N$, so $a^2x \in N$ or, $b^2x \in N$ or $a^2b^2 \in (N:M)$. Thus N is approximately 2- absorbing sub-module of M.

The following example shows that the opposite of Proposition 2.3 may not be true in general.

Example 2.4: Let ℝ be the field of real number, consider an ℝ-module is ℝ[x]. Let 0 be the zero polynomial. Thus N = {0} is prime and hence it is approximately 2- absorbing, sub-module, but it is not $Endo$- approximately 2- absorbing sub-module. Since there is f_1, f_2 where $f_1, f_2 : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by:
$f_1\ (a_0+a_1x+\ldots+a_nx^n) = a_1x + a_2x^2 + \cdots a_nx^n$.
$f_2\ (a_0+a_1x+\ldots+a_nx^n) = a_0 + a_2x^2 + \cdots a_nx^n$.
And let h(x) = 1+x ∈ ℝ[x], then ${f_1}^2(1+x) = x \neq$ zero polynomial, ${f_2}^2\ (1+x) = 1 \neq 0$ polynomial. However, ${f_1}^2{f_2}^2(1+x) = f_1\ (1) = 0$ = zero polynomial
Also, ${f_1}^2{f_2}^2\ (\mathbb{R}[x]) \nsubseteq N=\{0\}$ since ${f_1}^2{f_2}^2\ (2+x+3x) = f_1\ (2+3x) = 3x \neq 0$ polynomial. So, ${f_1}^2\ (1+x) \notin N$ and ${f_2}^2\ (1+x) \notin N$ as well as ${f_1}^2\ {f_2}^2\ (\mathbb{R}[x]) \nsubseteq N$ .Thus N is not $Endo$-approximately 2- absorbing sub-module of M.
The converse of Proposition 2.3 hold in the class of scalar modules.

Proposition 2.5: Let N be a proper sub-module of a scalar R- module M. Then N is approximately 2- absorbing sub-module if and only if N is an $Endo$- approximately 2-absorbing sub-module.

Proof:(⇐) It is follows by Proposition 2.3.
(⇒) Where $f_1, f_2 \in End$M, x∈ M , $f_1f_2(x) \in N$ and M is a scalar module. Then there exists a, b ∈R such that $f_1(x) = ax$, $f_2(x) = bx$ for all x ∈ M, $f_1\ f_2\ (x) = abx \in N$ which implies that either $a^2x \in N$ or, $b^2x \in N$ or $a^2b^2M \subseteq N$, since N approximately 2- absorbing sub-module. So, ${f_1}^2(x) \in N$ or, ${f_2}^2(x) \in N$, ${f_1}^2{f_2}^2(M) \subseteq N$ .Thus N is $Endo$- approximately 2- absorbing.

Corollary 2.6: If N is a proper sub- module of a finitely generated multiplication R- module M. Then N is an approximately 2- absorbing module if and only if N $Endo$-approximately 2-absorbing sub-module.

proof: Via [8] each finitely generated multiplication R- module M is a scalar module and so by Proposition 2.5, we get the result.
The following example shows that, $Endo$- approximately 2- absorbing sub- module does not imply $Endo$-prime sub- module in general.

Example 2.7: Take Z_{15} as Z-module. N = $(\bar{0})$ is an approximately 2- absorbing sub- module. Since Z_{15} is a finitely generated multiplication Z-module, it follows from Corollary 2.6 that N is an $Endo$- approximately 2- absorbing sub-module. Let f(x)=5x for any x in Z_{15}, thus f(3)∈ N, but 3 ∉ N and $f(Z_{15})=5Z_{15} \nsubseteq N$. Hence, N is not $Endo$-prime.

Proposition 2.8: Let R be an integral domain. If M is a divisible torsion non-zero module over R, then M has no $Endo$- approximately 2- absorbing sub-module.

Proof: Assume that M has an approximately 2- absorbing sub-module say K . Since M is a torsion module. So, an R-module $\frac{M}{K}$ is torsion module. Now, we will prove that $\frac{M}{K}$ is a torsion free R- module. Let $0 \neq m + K \in \frac{M}{K}$ Presume r(m+K) = K for some r ∈ R if r≠ 0, then rm

$\in$K and m$\notin$ K. The divisibility of M implies that M =rM, so there is $m_1 \in M$ such that m=rm_1. So, rm=r^2m_1 $\in$K. But K is approximately 2-absorbing submodule. Therefore, r^2m_1 $\in$K or $r^4 \in (K{:}M)$, if $r^2m_1 \in K$ then m$\in$ K which is a contradiction. If $r^4 \in (K{:}M)$ then $r^4M \subseteq K$. The divisibility of M implies that $r^4M = M$. Thus $M = K$which is a contradiction. Hence, r=0 and $\frac{M}{K}$ is torsion and torsion free module that implies $\frac{M}{K} = 0$. So, $M = K$ which is a contradiction. Hence, M has no approximately 2- absorbing sub- module. So, M has no $Endo$-approximately 2- absorbing sub- module.

Proposition 2.9: Let N be a proper sub- module of an R- module M, and K be a fully - invariant sub- module of R- module M such that contained in N. If $\frac{N}{K}$ is an $Endo$-approximately 2- absorbing sub-module of $\frac{M}{K}$. Then N is an $Endo$-approximately 2- absorbing sub-module of M.

Proof: Let f, h $\in$ EndM, m$\in$ M such that f∘h(m) $\in$ N. Define $f_1, h_1{:}\frac{M}{K} \longrightarrow \frac{M}{K}$ by f_1(m+K) = f(m)+K ; h_1(m+K) = h(m)+K, for all m$\in$ M. Clearly, f_1, h_1 are well defined. Now, $f_1 \circ h_1$(m+K) = $f_1(h_1$(m+K))= (h(m)+K)=(f∘h)(m)+K $\in$ ${}^{N}/_{K}$. So, $\frac{N}{K}$ is an EATA sub- module of ${}^{M}/_{K}$. Then, whichever h_1^2(m+K) $\in \frac{N}{K}$ or f_1^2 (m+K) $\in$ ${}^{N}/_{K}$ or f_1^2 ($h_1^2\left(\frac{M}{K}\right)) \subseteq {}^{N}/_{K}$. So, h_1^2(m)+K $\in \frac{N}{K}$ or f_1^2(m)+K $\in \frac{N}{K}$ or f_1^2 (h_1^2 (${}^{M}/_{K}$))$\subseteq \frac{N}{K}$.Thus h^2(m)$\in$ N or f^2(m) $\in$ N or $f^2(h^2(M)$ $\subseteq$ N. Thus N is an $Endo$-approximately 2- absorbing sub- module of M.

Now we introduce the concept of EATA module:

Definition 2.10: A non-zero R- module M is called EATA module, if (0) is an $Endo$-approximately 2-absorbing sub-module of M that is for all f, g $\in$ EndM, (f∘g)(m) =0 impels that $f^2(m) = 0$ or, $g^2(m) = 0$, $(f^2 \circ g^2)(M) = 0$.

Now we shall give some properties of EATA modules.

Remarks 2.11:

(1) By Proposition 2.5 and Corollary 2.6, we have M is a scalar or (finitely generated multiplication) R- module. Then M is an approximately 2- absorbing module if and only if M is EATA module.

(2) It is clearly that every $Endo$-prime module (quasi Dedekind) is EATA module. But the converse may not be true. See Example 2.7

(3) M is EATA module if and only if M is approximately 2- absorbing module over E= EndM.

Proposition 2.12: M is an approximately 2- absorbing module if and only if ann_M (f) is approximately 2- absorbing sub- module, for all f $\in$ EndM, such that f(M)≠0

Proof: (⇒)Let abm $\in$ ann_M (f). Then $f(abm) = 0$. So $abf(m) = 0$.

Since M is approximately 2- absorbing module, then either $a^2f(m) = 0$ or $b^2f(m) = 0$, $(ab)^2 \in$ annM, so $a^2m \in ann_M$ (f) or $b^2m \in ann_M$ (f) or $(ab)^2$M=(0) $\subseteq ann_M$ (f).

(⇐) Let abm $= 0$ whrere $a, b \in R, m \in M$. Take f= the identity mapping. So, m= i(m). Thus, $abi(m) = 0$ that is abm $\in ann_M$ (i). So $a^2m \in ann_M$ (i) or $b^2m \in ann_M$ (i) or $(ab)^2 \in (ann_M$ (i) : M). Thus i(a^2m) = a^2m = 0 or i(b^2m) = b^2m= 0 or $(ab)^2M \subseteq ann_M$ (i) So, i(a^2m) = a^2m = 0 or, i(b^2m) = b^2m= 0 or i$(ab)^2M = (ab)^2M = 0$. Hence i(a^2m) = a^2m = 0 or i(b^2m) = b^2m=0 or $(ab)^2 \in ann_M$.

Lemma 2.13 : Let M be approximately 2-absorbing R- module. The $annN$ is approximately 2-absorbing ideal in R for each $0 \neq N \leq M$.
Proof: Since $0 \neq N, ann\mathrm{N} < \mathrm{R}$, because if $annN = R,\ 1 \in annN$ and so $N = (0)$ which is a contradiction. Let $abc \in annN,$ for some $a, b, c \in R$. Then $ab(cN) = 0$. Since (0) is approximately 2- absorbing then $a^2cN = 0$ *or* $b^2cN = 0$ *or* $(ab)^2 \in (0, \mathrm{M}) = ann\mathrm{M} \leq ann\mathrm{N}$.So $a^2c \in ann\mathrm{N}$ *or* $b^2c \in ann\mathrm{N}$ *or* $(ab)^2 \in annN$. Thus $annN$ is approximately 2-absorbing.

Proposition 2.14: Let M be an EATA module, N sud-module M. Then $\ell\text{-ann}_{\mathrm{E}}\mathrm{N}$ is approximately 2- absorbing ideal of E, where E= EndM.

Proof: It follows by Remarks 2.11(3) and Lemma 2.13.

Proposition 2.15: Let $f : M \to M$ be an R-homomorphism If N is fully -invariant EATA sub- module of M such that f(M)$\not\subseteq$ N. Then f^{-1} (N) is also EATA sub- module of M. Provides EndM is commutative.

Proof: Let h, g $\in End$M , m ∈ M s.t (g∘h)(m) ∈ f^{-1} (N) then f(g∘h) ∈ N. Since N is EATA of M .
Then whichever $g^2(f(m)) \in N$ or $h^2(f(m)) \in N$ or $g^2 \circ h^2(f(M)) \subseteq N$;

Case(1) if $g^2(f(m)) \in N$, then f $g^2(m) \in N$, thus $g^2(m) \in f^{-1}$ (N).

Case (2) if $h^2(f(m)) \in N$, then f $h^2(m) \in N$ thus $h^2(m) \in f^{-1}$ (N).

Case(3) if $g^2 \circ h^2(f(M) \subseteq N$ then $f(g^2 \circ h^2(M)) \subseteq N$
so$(g^2 \circ h^2(M)) \subseteq f^{-1}$ (N). Thus f^{-1} (N) is an $Endo$-approximately 2-absorbing submodule of M .

Corollary 2.16: Let M be a fully stable, let f :M → M be an R-homomorphism. If N is EATA, then f^{-1} (N) is an $Endo$-approximately 2- absorbing sub-module of M .

proof: Since M is a fully stable, then EndM is commutative. Hence, the result by Proposition 2.15.

Proposition 2.17: Let f :M → M be an R-epimorphism. If N is
fully-invariant $Endo$-approximately 2- absorbing sub- module of M and kerf⊆ N. Then f(N) is EATA sub-module, provided EndM is commutative.

proof: Let h, g ∈ EndM, m´ ∈ M such that (g∘h)(m´) ∈f(N),since f is epimorphism then $m' = f(m)$ for some $m \in$ M so
$(g \circ h)f(m) \in f(N)$ we get $f(g \circ h)(m) \in f(N)$ since EndM is commutative. Also, $(g \circ h)(m) \in N$ since kerf⊆ N and N is $Endo$-approximately 2- absorbing of M, we have
$g^2(m) \in N$ or $h^2(m) \in N$ or $g^2 \circ h^2(M) \subseteq N$;

Case(1) if $g^2(m) \in N$, thus $f(g^2(m)) \in f(N)$, so $g^2f(m) \in f(N)$.Thus, $g^2(m') \in f(N)$.

Case (2) if $h^2(m) \in N$, then $f(h^2(m) \in f(N)$ thus
$h^2f(m) \in f(N)$. Since EndM is commutative thus $h^2(m') \in f(N)$.

Case (3) if $g^2 \circ h^2(M) \subseteq N$, then $f(g^2 \circ h^2(M) \subseteq f(N)$. So, $g^2 \circ h^2(f(M) \subseteq f(N)$ as EndM is commutative then $g^2 \circ h^2(M) \subseteq f(N)$ Thus f (N) is an $Endo$- approximately 2- absorbing sub- moduleM.

Proposition 2.18: Let N be an EATA sub- module of an R-module M and A be a sub-module of M which is M – injective sub- module. Then either $A \subseteq N$ or $A \cap N$ an EATA sub-module of A.

Proof: Presume $A \nsubseteq N$, then $A \cap N \subsetneq A$. Let f,g $\in End$A, x$\in$ A,
$f \circ g(x) \in A \cap N$. Presume $g^2(x) \notin A \cap N$, then $g^2(x) \notin N$. Now we are trying to prove that $f^2(x) \in A \cap N$ or $f^2 \circ g^2(A) \subseteq A \cap N$.
Since A is M –injective submodule, so there is h, k: $M \to A$ such that
h∘i=f, k∘i=g where i is the inclusion mapping. Clearly that
where i is the inclusion mapping and h, k $\in End$M.
But
$(f \circ g)(x) = [(h \circ i)\ (k \circ i)](x) = (h \circ i \circ k)(x) = (h \circ i)(k(x) = (h \circ k)(x)) \in N$.
Since N is an EATA sub-module of M $g^2(x) \notin N$, and implies $k^2(x) \notin N$, then either $h^2(x) \in N$ or, $h^2 \circ k^2(M) \subseteq N$. If $h^2(x) \in N$, then $h^2(x) \in A \cap N$ since $h^2(x) \in A$. If $h^2 \circ k^2(M) \subseteq N$ As $f^2 \circ g^2(A) = [(h^2 \circ i)\ (k \circ i)](A) = (h^2 \circ i)(k(A) = (h^2 \circ k^2)(A) \subseteq N$. Also, $f^2 \circ g^2(A) \subseteq A$, $(f \circ g)(A) \subseteq A \cap N$. Thus $A \cap N$ EATA sub- module of A.

Conclusions*:*

In this work, $Endo$-approximately 2-absorbing sub- module as a generalization of $Endo$ 2-absorbing sub-module is defined and studied. In addition, some of their properties and characterizations are described. Finally, some examples and significant results are given.

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