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Some Results on Essentially Quasi-DedekindModules

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Abstract

In this paper we give many connections between essentially quasi-Dedekind (quasi-Dedekind) modules and other modules such that Baer modules, retractable modules, essentially retractable modules, compressible modules and essentially compressible modules where an *R*-module *M* is called essentially quasi-Dedekind (resp. quasi-Dedekind) if, $Hom(M/N, M) = 0$ for all $N \leq_e M$ (resp. $N \leq M$). Equivalently, a module *M* is essentially quasi-Dedekind (resp. quasi-Dedekind) if, for each $f \in End_R(M)$, $Kerf \leq_e M$ implies $f = 0$ (resp. $f \neq 0$ implies $\ker f = 0$).

Keywords: Essentially quasi-Dedekind modules, Baer modules, retractable modules, compressible modules, monoform modules.

بعض النتائج للمقاسات شبه-الديديكاندية الواسعة

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الخلاصة:

نحن أعطينا العديد من العلاقات بين المقاسات شبه الديديكاندية الواسعة (المقاسات شبه الديديكاندية) وأنواع أخرى من المقاسات ، كمثال : مقاسات Baer ، المقاسات القابلة للانسحاب، المقاسات القابلة للانسحاب الواسعة، المقاسات القابلة للانضغاط و المقاسات القابلة للانضغاط الواسعة ، حيث يسمى المقاس M على الحلقة R مقاس شبه- ديديكاندي واسع (شبه ديديكاندي) إذا كان $Hom(M/N, M) = 0$ لكل مقاس جزئي واسع N من M (لكل مقاس جزئي N من M، على التوالي). يكافىء، المقاس M يُدعى مقاس شبه

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ديديكاندي واسع (مقاس ديديكاندي، على التوالي) اذا كان لكل $f \in End_R(M)$ ، $Kerf \leq_e M$ يؤدي الى $\ker f = 0$ ، على التوالي).

1. Introduction

In this work, R is a commutative ring with unity and M be an R-module. Recall that a submodule$N$ of M, N is called essential in M ($N \leq_e M$) if whenever $N \cap W = (0)$ implies $W = (0)$.

A submodule$N$ of M is called a direct summand ($N \leq^{\oplus} M$) if there exists a submodule$W$ of M such that $N \oplus W = M$ [1]. An R-module M is called retractable (resp. essentially retractable) if $Hom(M, N) \neq 0$, for all $0 \neq N \leq M$ (resp. $Hom(M, N) \neq 0$, for all $N \leq_e M$). The left annihilator of $N \subseteq M$ in $S = End_R(M)$ (denoted by $L_S(N)$) is the setoff all elements g $\in S$ such that $\phi N = 0$. The right annihilator of $T \subseteq S$ in M is denoted by $r_M(T)$ is the set of all elements $m \in M$ such that $Tm = 0$, [2]. An R-module M is called prime if $ann_R M = ann_R N$ for all nonzero submodule$N$ of M. An R-module M is called essentially prime if $ann_R M = ann_R N$ for each $N \leq_e M$ [3]. Recall that an R-module is called quasi-Dedekind if $Hom(M/N, N) = 0$ for all $N \leq M$, [4]. Equivalently M is quasi-Dedekind, if for each $f \in End_R(M)$, $f \neq 0$ implies ker$f = 0$, [4]. Tha'ar in [3] gave the following: an R-module M is called essentially quasi-Dedekind R-module if $Hom(M/N, N) = 0$ for all $N \leq_e M$. Equivalently, M is essentially quasi-Dedekind, if for each $f \in End_R(M)$, $Ker f \leq_e M$ implies $f = 0$, [3]. An R-module M is called Baer if for all $N \leq M$, $L_S(N) = Se$, with $e^2 = e \in S = End_R(M)$, [2]. Equivalently, M is Baer if, for all ideals I in S, $r_M(I) = eM$with $e^2 = e \in S = End_R(M)$, [2, p.10].

Remarks and Examples(1.1):

1) Every semisimple$R$-module is a Baer R-module, [2 , Ex 2.1.2]
2) Z as a Z-module is Baer, but $Z^n = Z \oplus Z \oplus Z \ldots$ as a Z-module is not Baer, [2, Ex 2.4.2] .
3) $Z^n = \underbrace{Z \oplus Z \oplus Z \ldots \oplus Z}_{\text{n-times}}$ is a Baer Z-module for all $n \in N$, [2 , Ex 2.1.2] .
4) The Z-module $M = Z \oplus Z_2$ is not Baer, even though Z andZ_2 are both Baer Z-modules, [2, Ex 2.4.3] .
5) Q as a Z-module is not retractable, [2, p.44] .
6) A direct summand of a retractable module, need not be retractable, for example: $Z_{p^\infty} \oplus Z$ is a retractable as Z-module, but Z_{p^∞} is not retractable as a Z-module, [5, p.356].

The following proposition was given in [2, Lemma 2.2.5]. However, a different proof is given here.

Proposition(1.2): Every Baer R-module is an essentially quasi-Dedekind (K-nonsingular) R-module.

Proof : Suppose that M is a Baer R-module, let $f \in End_R(M)$, $f \neq 0$. To prove that $Ker f \not\leq_e M$ .If $Ker f = 0$ then M is a quasi-Dedekind R-module, so it is an essentially quasi-Dedekind R-module. If $Kerf \neq 0$,since$M$ is Baer, then by [6, Th 1.5] , $Kerf \leq^{\oplus} M$ .Thus $Kerf \not\leq_e M$. Thus M is an essentially quasi-Dedekind R-module. □

The converse of (Prop 1.2) is not true in general, as the following example shows.

Example(1.3): It is well-known that by (Rem.and.Ex 1.1(2)), $M = Z \oplus Z \oplus Z ...$ as a Z-module is not Baer. But Z is an essentially quasi-dedekind$Z$-module; that is Z is an essentially quasi-Dedekind relative to Z. Then by [3, Th 1.3.5], $M = Z \oplus Z \oplus Z ...$ is an essentially quasi-Dedekind Z-module.

Corollary(1.4): If M is a Baer R-module, then $End_R(M)$ is an essentially quasi-Dedekind ring .
Proof : Since M is a Baer R-module, so by [2, Th 4.1.1], $End_R(M)$ is a Baer ring. Thus by (Prop 1.2), $End_R(M)$ is an essentially quasi-Dedekind ring. □

Corollary(1.5): Let M be a retractable R-module. If $End_R(M)$ is a Baer ring, then M is an essentially quasi-Dedekind R-module.
Proof : Suppose that $End_R(M)$ is a Baer ring, and since M is a retractable R-module, so by [7 ,Prop 4.1.4], M is a Baer R-module. Thus by (Prop 1.2), M is an essentially quasi-Dedekind R-module. □

The following proposition given in [6, Prop 3.6], we give the details of the proof.

Proposition(1.6): Let M be a retractable R-module. If M is an essentially quasi-Dedekind R-module, then $S = End_R(M)$ is a right nonsingular ring and hence essentially quasi-Dedekind.
Proof : Suppose that M is an essentially quasi-Dedekind R-module. To prove that S_s is a nonsingular ring, where $S = End_R(M)$, we must prove $\{\phi \in S : r_S(\phi) \leq_e S_s\} = 0$. Let $\phi \in S$ such that $r_S(\phi) \leq_e S_s$ .If $\phi = 0$, nothing to prove. If $\phi \neq 0$, then $Ker\phi \nleq_e M$. Hence there exists $0 \neq N \leq M$, N is a relative complement to $Ker\phi$, and so $N \cap Ker\phi = 0$. By retractability of M, there exists $\psi : M \longrightarrow N$, $\psi \neq 0$. Consider the following : $M \xrightarrow{\psi} N \xrightarrow{i} M \xrightarrow{\phi} M$, where i is the inclusion mapping. $\phi o i o \psi \neq 0$, to show this: assume $\phi o i o \psi = 0$. Since $\psi \neq 0$, there exists $m \in M$ such that $\psi(m) = n \neq 0$, $n \in N$, hence $0 = \phi o i o \psi(m) = (\phi o i)(n) = \phi(n)$ which implies $n \in Ker\phi$, hence $n \in N \cap Ker\phi = 0$ which is a contradiction.Thus $\phi o i o \psi \neq 0$, so $\phi o \psi \neq 0$, implies $\psi \notin r_S(\phi)$, then $0 \neq \psi s \not\subseteq r_S(\phi)$.
We claim that $\psi s \cap r_S(\phi) = 0$. Suppose that, there exists $g \in \psi s$ and $g \in r_S(\phi)$, then $g = \psi o h$ and $\phi o g = 0$ for some $g, h \in S = End_R(M)$, so $g(M) = \psi o h(M) \subseteq \psi(M) \subseteq N$ which implies $g(M) \subseteq N$, also $\phi o g(M) = 0$,then $\phi(g(M)) = 0$ implies $g(M)) \subseteq Ker\phi$. Thus $g(M)) \subseteq N \cap Ker\phi = 0$; that is $g = 0$. But this contradicts the essentiality of $r_S(\phi)$. Therefore $Ker\phi \leq_e M$ and hence $\phi = 0$, since M is an essentially quasi-Dedekind R-module. Thus $S = End_R(M)$ is a right nonsingular ring and hence essentially quasi-Dedekind. □

We prove the following proposition:

Proposition(1.7): Let M be a uniform R-module. The following statements are equivalent:
1) M is an essentially quasi-Dedekind R-module.
2) M is a Baer R-module.
3) M is a quasi-Dedekind R-module.

4) M is a prime R-module.
5) M is an essentially prime R-module.

Proof :(1) $\Rightarrow$ (2): Since M is a uniform R-module, so by [4, Prop 2.1.1], M is an extending R-module. But M is an essentially quasi-Dedekindextending R-module, implies M is a Baer R-module , by [7, lemma 2.2.4].
(2) $\Rightarrow$ (3): Since M is a uniform R-module, so by [8, Prop 2.1.1] ,M is an indecomposable R-module. But M is a Baer and indecomposable R-module, implies M is a quasi-Dedekind R-module, by [2, Th 2.3.5].
(3) $\Rightarrow$ (1) : It follows by [3, Rem.and.Ex 1.2.2(1)] .
(3) $\Leftrightarrow$ (4) : It follows by [3, Th 0.2.16] .
(4) $\Leftrightarrow$ (5) : It is clear. □

To give the next result, we prove the following lemma.

Lemma(1.8): Let Mbe an R-module. If M is a uniform R-module, then $E(M)$ is a uniform R-module .

Proof: Let$U \le E(M)$, $U \ne 0$. To prove $U \cap W \ne 0$ for all $0 \ne W \le E(M)$ .Since $M \le_e E(M)$, then $U \cap M \ne 0$, $W \cap M \ne 0$. But since$U \cap M \le M$, $W \cap M \le M$ and M is uniform, so that $(U \cap M) \cap (W \cap M) \ne 0$. This implies$(U \cap W) \cap M \ne 0$, hence $U \cap W \ne 0$. Thus $E(M)$ is a uniform R-module. □

Proposition(1.9): Let M be a uniform R-module with $ann_R(M) = ann_R(E(M))$. Then the following statements are equivalent:

1)$E(M)$ is an essentially quasi-Dedekind R-module.

2)$E(M)$ is a Baer R-module.

3) $E(M)$ is a quasi-Dedekind R-module.

4) M is a quasi-Dedekind R-module.

5) M is an essentially quasi-Dedekind R-module.

Proof : (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) : It follows by (Lemma 1.8) and (Prop 1.7).

(3) $\Leftrightarrow$ (4) : It follows by Lemma 1.8 and [3, Coro 0.2.18].

(4) $\Leftrightarrow$ (5) : It is clear. □

Recall that a nonzero R-module M is compressible (resp. essentially compressible) if, M can be embedded in each of its nonzero submodule (resp. in each essential submodule), see [9], [10]. It is clear that every compressible module is an essentially compressible module. Recall that an R-module M is monoform if, for each $N \le M$ and for each $f \in Hom(N, M)$, $f \ne 0$ implies $Kerf = 0$, [9]. A module M is called polyform if, for all $f \in Hom(N,M)$, $f \neq 0$, for all $N \leq M$ implies $Kerf \not\leq_e N$, [10]. It is clear that every monoform module is a polyform module.

P.F.Smith in [9 , Coro 2.5], prove the following proposition.

Proposition(1.10): Every compressible *R*-module is a monoform*R*-module, and hence a quasi-Dedekind *R*-module.

The converse of (Prop 1.10), is not true in general, for example :The*Z*-module *Q* is uniform and prime, hence it is monoform [3, Prop 2.3.19], but it is not compressible, since $Hom(Q,Z)=0$; that is *Q*can not be embedded in *Z*.

The converse of (Prop 1.10) holds whenever M is finitely generated.

Proposition(1.11): Let *M* be a finitely generated *R*-module. Then *M* is compressible if and only if *M* is monoform.

Proof : $\Rightarrow$) It is clear by (Prop 1.10).

$\Leftarrow$) By [11, Th 2.3], *M* is a uniform prime *R*-module. But *M* is finitely generated, so by [11, Lemma 1.9], *M* is compressible. □

The condition *M* is finitely generated can not be dropped from (Prop 1.11). For example: The *Z*-module *Q* is monoform, but it is not compressible. In fact *Q* is not finitely generated.

Recall that an R-module M is called essentially prime if, $ann_R(M)=ann_R(N)$ for all $N \leq_e M$, [3]. It is clear that every prime module is an essentially prime module.

Corollary(1.12): Let *M* be a finitely generated *R*-module. The following statements are equivalent:
1) *M* is a monoform*R*-module.
2) *M* is a uniform prime (uniform essentially prime) *R*-module.
3) *M* is a uniform quasi-Dedekind (uniform essentially quasi-Dedekind) *R*-module.
4) *M* is a compressible *R*-module.

Proof :(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) : It follows by [3, Prop 2.3.19].
(1) $\Leftrightarrow$ (4) : It follows by (Prop 1.11). □

Corollary(1.13): Let *M* be a finitely generated faithful *R*-module. The following statements are equivalent:
1) *M* is a compressible *R*-module.
2) *M* is a monoform *R*-module.
3) *M* is a uniform prime (uniform essentially prime) *R*-module.
4) *M* is a uniform quasi-Dedekind (uniform essentially quasi-Dedekind) *R*-module.
5) *M* is a uniform nonsingular *R*-module.

proof : It follows by (Coro 1.12) and [3, Prop 2.3.21]. □

Also (as we mention before) it is clear that every monoform module is a quasi-Dedekind module. However the inverse implication holds under the class of retractable modules.

Proposition(1.14): Let *M* be a retractable *R*-module. Then *M* is a quasi-Dedekind *R*-module if and only if *M* is a monoform*R*-module.

Proof :$\Rightarrow$) Since *M* is a quasi-Dedekind *R*-module, so by [3, Th 0.2.6], for each $f \in End_R(M)$, $f \neq 0$ implies *Kerf* = 0. Hence by [7, Prop 1.2], *M* is a compressible *R*-module, and then by (Prop 1.10), *M* is a monoform*R*-module.

$\Leftarrow$) It is clear. □

Recall that a compressible module is critically compressible if, it can not be embedded in any proper factor module [7]. However, compressible and critically compressible modules are equivalent under the class of modules over duo ring, where a ring *R* is called duo if every left (right) ideal of *R* is two sided ideal. Thus compressible and critically compressible modules are equivalent under the class of modules over commutative ring. Hence (Prop 1.3) in [7], can be restated as follows "Let *M* be a retractable module. Then *M* is compressible if and only if *M* is monoform".

Hence by combining (Prop 1.14) and (Prop 1.3) in [7], we get the following corollary.

Corollary(1.15): Let*M* be a retractable *R*-module. Then the following statements are equivalent:

1) *M* is a compressible *R*-module.
2) *M* is a monoform*R*-module.
3) *M* is a quasi-Dedekind *R*-module.

By using [3,Prop 0.2.6], then (Th 1.4) in [7] can be restated as follows:

Let *M* be an *R*-module. The following statements are equivalent:

1) *M* is compressible and quasi-Dedekind.
2) *M* is compressible and $End_R(M)$ is a domain.
3) *M* is retractable and quasi-Dedekind.
4) *M* is retractable and $End_R(M)$ is domain.

Note that by using (Prop 1.10), the condition *M* is quasi-Dedekind can be dropped from (1).

However we have the following corollary:

Corollary(1.16): Let *M* be a retractable *R*-module. The following statements are equivalent:

1) *M* is a compressible *R*-module.
2) *M* is a monoform *R*-module.
3) *M* is a quasi-Dedekind *R*-module.
4) *M* is a uniform polyform*R*-module.
5) *M* is a uniform quasi-Dedekind (uniform essentially quasi-Dedekind) *R*-module.
6) $End_R(M)$ is a domain ($End_R(M)$ has no zero divisors).

Proof :(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) : It follows by (Coro 1.15).

(2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) : It follows by [3, Prop 2.3.19].

(1) $\Leftrightarrow$ (6) : It follows by [7, Th 1.4]. □

Corollary(1.17): Let M be a uniform retractable R-module. If M is nonsingular then M is compressible.
Proof : It is clear that every uniform nonsingular is monoform. Hence the result follows by (Coro 1.15). □

Corollary(1.18): Let$M$ be a uniform retractable R-module. Then M is compressible if and only if M is polyform.
Proof : $\Rightarrow$) It follows by (Prop 1.10).
$\Leftarrow$) By [3, Prop 2.3.19], M is monoform, and by (Coro 1.15), M is compressible. □

Recall that an R-module M is said to satisfy (*) if, for each nonzero submodule$N$ of M, $ann_R(M/N) \not\subseteq ann_R(M)$, [8] .Ahmed A.A in [12] proved that " Every module satisfies (*), then $Hom(M,N) \neq 0$, for all nonzero submodules$N$ of M; that is M is retractable".

Hence we conclude the following.

Corollary(1.19): Let M be an R-module satisfies (*) and $End_R(M)$ is a commutative ring. The following statements are equivalent:
1) $R/ann_R(M)$ is an integral domain.
2) $ann_R(M)$ is a prime ideal of R.
3) M is a prime R-module.
4) M is a compressible R-module.
5) M is a monoform$R$-module.
6) M is a quasi-Dedekind R-module.
7) $End_R(M)$ is an integral domain.
8) M is a rational extension of N, for all $N \leq M$ (i.eM is strongly uniform).
Proof : Since M satisfies (*) , M is retractable. Hence by (Coro 1.16), we get
(4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) .
(1) $\Leftrightarrow$ (2) : It is clear.
(2) $\Leftrightarrow$ (3) : It follows by [12, Prop 1.9].
(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) : It follows by [12, Th 2.5]. □

Now we ask the following questions:

(1) What is the relationships between an essentially compressible module and an essentially quasi-Dedekind module.
(2)What is the relationships between an essentially compressible module and a monoform module.
(3) What is the relationships between an essentially compressible module and a polyform module.

For the 1st question, we claim that an essentially compressible module implies an essentially quasi-Dedekind module. However, we can not prove this and we can not disprove. But, the next proposition shows that every an essentially compressible module is an essentially prime module.

Proposition(1.20): Let M be an essentially compressible R-module, then M is an essentially prime R-module.

Proof : Let $N \leq_e M$. To prove $ann_R(N) = ann_R(M)$. Let $A = ann_R(N)$,so $AN = 0$. Since M is essentially compressible, then there exists a monomorphism $f: M \longrightarrow N$. Hence $f(AM) = Af(M) \subseteq AN = 0$. This implies $AM = 0$, since f is monomorphism. So that $A \subseteq ann_R(M)$; that is $ann_R(N) \subseteq ann_R(M)$. Thus $ann_R(N) = ann_R(M)$. □

To answer the 2nd question, first we have.

Lemma(1.21): Let M be an R-module .Then M is monoform if and only if for each $N \leq_e M$ and for each nonzero $f \in Hom(N, M)$ implies f is monomorphism.

Proof: $\Rightarrow$) It is clear.

$\Leftarrow$) Let $N \leq M$. If $N \leq_e M$ then we have nothing to prove. Assume that $N \not\leq_e M$ and let $f: N \longrightarrow M$ such that $f \neq 0$. Since $N \not\leq_e M$ implies there exists $K \leq M$ (relative complement of N) such that $N \oplus K \leq_e M$. Define $g: N \oplus K \longrightarrow M$ by $g(n + k) = f(n)$ for all $n+k \in N \oplus K$. It is clear that g is well-defined, $g \neq 0$ and $K \subseteq Ker g$. But g is monomorfphism by hypothesis, hence $K = 0$. Thus our assumption $N \not\leq_e M$ is false. Thus M ismonoform. □

Recall that an R-module M is essentially retractable if , $Hom(M, N) \neq 0$,for all $N \leq_e M$, [10]. It is clear that every retractable module is an essentially retractable module. Note that Z_4 as a Z-module is retractable, so it is essentially retractable .

Proposition(1.22): Let M be an essentially retractable R-module. If every essential submodule of M is quasi-Dedekind, then M is monoform.

Proof: Let $N \leq_e M$ and let $f: N \longrightarrow M$, $f \neq 0$. Since M is an essentially retractable R-module, then $Hom(M, N) \neq 0$ .So there exists $g: M \longrightarrow N$ and $g \neq 0$,thus $gof \in End_R(N)$. We claim that $gof \neq 0$. Since $iog \in End_R(M)$, where i is the inclusion mapping, then $iog \neq 0$. But M is quasi-Dedekind, hence iog is monomorphism, thus g is monomorphism. Hence, if $gof = 0$, then $g(f(N)) = 0$ implies $f(N) = 0$; that is $f = 0$ which is a contradiction. Thus $0 \neq gof \in End_R(N)$, hence gof is monomorphism, since N is quasi-Dedekind by hypothesis. This implies f is monomorphism and so by (Lemma 1.21), M is monoform. □

Before giving the next corollary, we have the following lemma.

Lemma(1.23): If M is an essentially compressible module, then M is a retractable module.

Proof: By[13,Th 3.1], for each nonzero submodule$N$ of M and for each endomorphism $f \in Hom(M, N)$, $f|_N \neq 0$. Thus $Hom(M, N) \neq 0$; that is M is a retractable module. □

Corollary(1.24): Let M be an essentially compressible R-module. If every essential submodule of M is quasi-Dedekind, then M is monoform.

Proof : By (Lemma 1.23), M is a retractable R-module and hence an essentially retractable R-module. Thus the result is obtained by (Prop 1.22). □

To give the next result, first we prove the following lemma.

Lemma(1.25): Let M be an R-module. Then M is polyform if and only if for each $N \leq_e M$ and for each nonzero $f \in Hom(N,M)$ implies $Kerf \not\leq_e N$.

Proof : $\Rightarrow$) It is clear.

$\Leftarrow$) Let $N \leq M$. If $N \leq_e M$ then we have nothing to prove.If $N \not\leq_e M$, let $f : N \longrightarrow M$ such that $f \neq 0$. Since $N \not\leq_e M$, then there exists $K \leq M$ (relative complement of N), hence $N \oplus K \leq_e M$. Define $g : N \oplus K \longrightarrow M$ by $g(n + k) = f(n)$ for all $n + k \in N \oplus K$. It is clear that g is well-defined, $g \neq 0$. Hence $Kerg \not\leq_e N \oplus K$ by hypothesis. But

$$Kerg = \{n + k : g(n + k) = 0, \quad n + k \in N \oplus K\}$$

$$= \{n + k : f(n) = 0, \quad n \in N, k \in K\} = \{n + k : n \in Kerf, k \in K\} = Kerf \oplus K .$$

Thus $Kerf \oplus K \not\leq_e N \oplus K$. Since $K \leq_e K$,then by [1, Coro 5.1.8, p.112], $Kerf \not\leq_e N$. Therefore M is a polyform$R$-module. □

We finish this paper by the following theorem.

Theorem(1.26): Let M be an essentially compressible R-module such that every essential submodule of M is an essentially quasi-Dedekind R-module, then M is a polyform$R$-module.

Proof : By(Lemma 1.25) it is enough to show that for each $N \leq_e M$ and $f : N \longrightarrow M$, $f \neq 0$ then $Kerf \not\leq_e N$. Since M is essentially compressible, then there exists $g : M \longrightarrow N$ such that g is monomorphism. Hence $gof \in End_R(N)$ and $gof \neq 0$ because if $gof = 0$ then $gof(N) = g(f(N)) = 0$ which implies $f(N) = 0$; that is $f = 0$ which is a contradiction. Thus $Ker(gof) \not\leq_e N$, since N is essentially quasi-Dedekind. But it is easy to check that $Kerf = Ker(gof)$. Therefore $Kerf \not\leq_e N$.□

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