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Certain Types of Weakly Closed Function

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Abstract

The purpose of this paper is to give the condition under which every weakly closed function is closed and to give the condition under which the concepts of weakly-semi closed function and weakly pre-closed function are equivalent. Moreover, characterizations and properties of weakly semi closed functions and weakly pre-closed function was given.

Keywords: weakly closed function, pre-closed set, G_δ set, n -preserving function.

انواع معينة من الدوال المغلقة الضعيفة

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الخلاصة:

والغرض من هذا البحث هو إعطاء الشروط التي بموجبها يمكن اعتبار كل دالة مغلقة ضعيفة دالة مغلقة وإعطاء الشروط التي تحتها يتكافئ مفهومي الدوال شبه المغلقة الضعيفة والدوال قبل المغلقة الضعيفة. وعلاوة على ذلك، أعطيت بعض الأوصاف والخصائص لمفهومي الدوال شبه المغلقة الضعيفة والدوال قبل المغلقة الضعيفة.

Introduction :

In this section, we recall some definitions needed in this work.

Definitions 1.1: Let A be a subset of a topological space (X, T)

1. A is said to be semi-open [1] set in X if $A \subseteq \text{Cl}(\text{Int}(A))$

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2. A is said to be semi-closed [1] set in X if $\text{Int}(\text{Cl}(A)) \subseteq A$
3. A is said to be pre-open [2] set in X if $A \subseteq \text{Int}(\text{Cl}(A))$
4. A is said to be pre-closed [2] set in X if $\text{Int}(\text{Cl}(\text{Int}(A))) \subseteq A$
5. A is said to be α -closed [3] set in X if $\text{Cl}(\text{Int}(\text{Cl}(A))) \subseteq A$

Recall that the intersection of all semi-closed sets containing A is called the semi-closure[2] of A and denoted by $S\text{-Cl}(A)$. The pre-closure of A is the intersection of all pre-closed sets containing A . The union of all semi-open sets contained in A is called the semi-interior [2] of A and denoted by $S\text{-Int}(A)$. Similarly, the pre-interior of A is the union of all pre-open sets contained in A and it is denoted by $p\text{-Int}(A)$. In [4] Maheshwari and Prasad introduced the concept of semi- T_2 -space, a topological space (X, T) is said to be semi- T_2 -space if each distinct points x, y in X there exists two distinct semi-open sets U and V such that $x \in U$ and $y \in V$. Similarly, we define pre- T_2 -space and α - T_2 -space. Levine [1] defined the semi-closed function a function $f: X \rightarrow Y$ is called semi-closed function if the image of each closed subset of X is a semi-closed subset of Y . Similarly, we define pre-closed function and α -closed function. Recall that in a topological space (X, T) a G_δ set [5] is a countable intersection of open sets, furthermore a topological space (X, T) is termed perfectly normal if it is normal and every closed subset is G_δ set. A subset A of a topological space (X, T) is called nowhere dense (in X) if there is no neighborhood in X on which A is dense. Jozef Doboš [6] introduced the concept of nd -preserving function, let $f: X \rightarrow f(X)$ be a function in a topological space (X, T) . We say that f is nowhere dense sets preserving (abbreviated henceforth as nd -preserving) if the image of a nowhere dense set in X is a nowhere dense set in $f(X)$.

2. Weakly Semi-Closed Functions

In this section we introduce and study the first type of weakly closed functions, namely semi-closed function. First we recall for some definitions and facts.

Definition 2.1 [7]: A function $f: X \rightarrow Y$ is called weakly closed function or simply WC- function iff $\text{Cl}(f(U)) \subseteq f(\text{Cl}(U))$ for each open set U of X .

It is known that (see [8]) a function $f: X \rightarrow Y$ is called closed function if $\text{Cl}(f(U)) \subseteq f(\text{Cl}(U))$ for each subset U of X . It is clear that every closed function is weakly closed function but the converse is not necessarily true unless we add an extra condition as in the following proposition.

Proposition 2.2: Let $f: X \rightarrow Y$ be a weakly Closed function and suppose that for closed subset F of X and for each y in $Y - f(F)$, there exists an open set U of X such that $F \subseteq U$ and $f^1(y) \cap U = \emptyset$, then f is closed function.

Proof: Let F be any closed set in X and y be any point in $Y - f(F)$, i.e. $f^1(y) \subseteq X - F$, hence there exists an open set U of X such that $F \subseteq U$ and $f^1(y) \cap U = \emptyset$, which implies that $\{y\} \cap f(U) = \emptyset$. Therefore $y \in Y - f(U) \subseteq Y - f(F)$. Since f is weakly closed function then $\text{Cl}(f(\text{Int}(F))) \subseteq f(F)$ which implies $y \in \text{Int}(Y - f(\text{Int}(F))) \subseteq Y - f(F)$, this means that y is an interior point of $Y - f(F)$, so $Y - f$ is an open set of Y , hence $f(F)$ is closed set of Y . Therefore f is a closed function ■

Theorem 2.3: A function $f: X \rightarrow Y$ is a semi-closed function iff $S\text{-Cl}(f(U)) \subseteq f(\text{Cl}(U))$ for each subset U of X .

Proof: suppose f is a semi-closed function and U is any subset of X . $f(\text{Cl}(U))$

is a semi-closed of Y and $f(U) \subseteq f(\text{Cl}(U))$, but $S\text{-Cl}(f(U))$ is the smallest semi-closed set in Y containing $f(U)$, therefore $S\text{-Cl}(f(U)) \subseteq f(\text{Cl}(U))$. Conversely, suppose $S\text{-Cl}(f(U)) \subseteq f(\text{Cl}(U))$ for each subset U of X . Let V be any closed subset of X implies $S\text{-Cl}(f(V)) \subseteq f(\text{Cl}(V))$, i.e. $f(V)$ is semi-closed set in Y . Thus f is semi-closed function ■

Corollary 2.4: If $f: X \rightarrow Y$ is weakly closed function with closed fibers then f is closed function.

Proof: Let F be a closed set in X and let $y \in Y - f(F)$ then $f^{-1}(y) \subseteq X - F$, implies that $f^{-1}(y)$ is a closed set in X . Let $U = f^{-1}(y)$, then U is open set in X , $F \subseteq U$ and $f^{-1}(y) \cap U = \emptyset$, so by theorem 2.3 f is closed function ■

Corollary 2.5: If $f: X \rightarrow Y$ is weakly closed, injective function and X is a T_1 -space then f is closed function.

Proof: for any point y in Y $f^{-1}(y) = \{x\}$ is closed in X because f is injective function and X is a T_1 -space. Therefore by corollary 2.4 f is closed function ■

Definition 2.6[7]: A function $f: X \rightarrow Y$ is called weakly semi-closed function or simply WSC-function iff $S\text{-Cl}(f(\text{Int}(F))) \subseteq f(F)$ for each closed set F in X .

It is very easy to see that every WC-function is WSC-function and every semi-closed function is WSC-function (Figure-1)

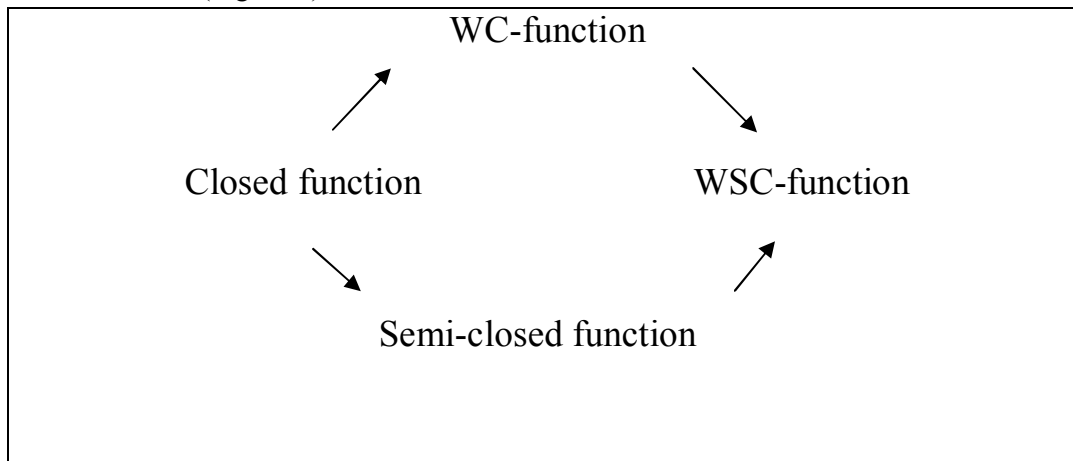


Figure1-

We are going to give an example of WSC-function which is not a semi-closed function

Example 2.7: Let $\{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$. Then the identity function $I_X: (X, \tau) \rightarrow (X, \sigma)$ is a WSC-function which is not semi-closed function since for $F = \{b, c\}$, $f(F)$ is not semi-closed set in (X, σ) .

The proof of the following theorem is straightforward,

Theorem 2.8[7]: For function $f: X \rightarrow Y$ the following are equivalent

- (i) f is WSC-function.
- (ii) For each closed subset F of X , $S\text{-Cl}(f(\text{Int}(F))) \subseteq f(\text{Cl}(\text{Int}(F)))$.
- (iii) For regular closed subset F of X $S\text{-Cl}(f(\text{Int}(F))) \subseteq f(\text{Cl}(\text{Int}(F)))$.
- (iv) For each pre-closed subset F of X $S\text{-Cl}(f(\text{Int}(F))) \subseteq f(F)$.
- (v) For each α -closed subset F of X $S\text{-Cl}(f(\text{Int}(F))) \subseteq f(F)$.

Now, we recall the definition of θ -closed set as follows.

Definition 2.9[7]: A subset A of a space X is called θ -closed set if $A = \theta\text{-Cl}(A)$, where $\theta\text{-Cl}(A) = \{x \in X : \text{Cl}(U) \cap A \neq \emptyset \forall \text{ open set } U \subseteq X \text{ with } x \in U\}$.

Notice that every θ -closed set is closed.

Theorem 2.10: The following statements are equivalent for a function $f: X \rightarrow Y$

- (i) f is WSC-function.
- (ii) For each regular open subset U of X $S\text{-Cl}(f(U)) \subseteq f(\text{Cl}(U))$
- (iii) For each $B \subseteq Y$ and for each open subset U of X with $f^{-1}(B) \subseteq U$, there exists a semi-open set W of Y with $B \subseteq W$ and $f^{-1}(W) \subseteq \text{Cl}(U)$
- (iv) For each $y \in Y$ and for each open set U of X with $f^{-1}(y) \subseteq U$ there exists a semi-open set W of Y with $y \in W$ and $f^{-1}(W) \subseteq \text{Cl}(U)$.
- (v) For each subset A of X $S\text{-Cl}(f(\text{Int}(\text{Cl}(A)))) \subseteq f(\text{Cl}(A))$.
- (vi) For each subset A of X $S\text{-Cl}(f(\text{Int}(\theta\text{-Cl}(A)))) \subseteq f(\theta\text{-Cl}(A))$.
- (vii) For each pre-open set A of X $S\text{-Cl}(f(A)) \subseteq f(\text{Cl}(A))$.

Proof: It is clear that (i) \rightarrow (ii), (i) \rightarrow (v) \rightarrow (vii) \rightarrow (i), (i) \rightarrow (vi) and (iii) \rightarrow (iv). To prove that (ii) \rightarrow (iii) let B be any subset of Y and U be any subset of X with $f^{-1}(B) \subseteq U$, then $f^{-1}(B) \cap \text{Cl}(X - \text{Cl}(U)) = \emptyset$ and consequently, $B \cap f(\text{Cl}(X - \text{Cl}(U))) = \emptyset$. Since $X - \text{Cl}(U)$ is a regular open set we have $B \cap S\text{-Cl}(X - \text{Cl}(U)) = \emptyset$ because $S\text{-Cl}(X - \text{Cl}(U)) \subseteq f(\text{Cl}(X - \text{Cl}(U)))$. Let $V = Y - [S\text{-Cl}(f(X - \text{Cl}(U)))]$, then V is semi-open set with $B \subseteq V$ and $f^{-1}(V) \subseteq X - f^{-1}(S\text{-Cl}(f(X - \text{Cl}(U)))) \subseteq X - f^{-1}(f(X - \text{Cl}(U))) \subseteq \text{Cl}(U)$.

Now, to prove that (iv) \rightarrow (i), let F be a closed set in X and let $y \in Y - f(F)$. Since $f^{-1}(y) \subseteq X - F$, there exists a semi-open set $W \subseteq Y$ with $y \in W$ and $f^{-1}(W) \subseteq \text{Cl}(X - F) = X - \text{Int}(F)$. Therefore $W \cap Y - S\text{-Cl}(\text{Int}(F)) = \emptyset$ this means $S\text{-Cl}(f(\text{Int}(F))) \subseteq f(\text{Int}(F))$ thus f is weakly semi-closed function.

We are going to prove that (iv) \rightarrow (vii). Notice that $\theta\text{-Cl}(A) = \text{Cl}(A)$ for each pre-open set A subset of X , so $S\text{-Cl}(f(A)) \subseteq S\text{-Cl}(f(\text{Int}(\text{Cl}(A)))) = S\text{-Cl}(f(\text{Int}(\theta\text{-Cl}(A)))) \subseteq f(\theta\text{-Cl}(A)) = f(\text{Cl}(A))$. This completes the proof of the theorem ■

3. Weakly pre-Closed Function

In this section we introduce and study the second type of weakly closed function, namely weakly pre-closed function. First we recall the following definition and facts

Definition 3.1: [2] A function $f: X \rightarrow Y$ is called pre-closed function iff $f(W)$ is closed for each closed set W in X .

Proposition 3.2: A function $f: X \rightarrow Y$ is a pre-closed function iff $\text{pre-Cl}(f(U)) \subseteq \text{Cl}(f(U))$ for each subset U of X .

(Recall that $\text{pre-Cl}(f(U))$ is the smallest pre-closed set of Y containing $f(U)$)

Proof: Suppose f is a pre-closed function and U is any subset of X then $f(\text{Cl}(U))$ is a pre-closed set in Y and contain $f(U)$ which implies $\text{pre-Cl}(f(U)) \subseteq f(\text{Cl}(U))$. Conversely, let U be any closed set in X since $U = \text{Cl}(U)$ then $\text{pre-Cl}(f(U)) \subseteq f(\text{Cl}(U)) = f(U)$ but $f(U) \subseteq \text{pre-Cl}(f(U))$. Therefore $f(U)$ is a pre-closed set in Y thus f is a pre-closed function ■

This motivates the definition of weakly pre-closed function as following.

Definition 3.3[8]: A function $f: X \rightarrow Y$ is called weakly pre-closed function iff $\text{pre-Cl}(f(\text{Int}(F))) \subseteq f(F)$ for every closed set F in X

It is very easy to see that every WC-function is weakly pre-closed function and every pre-closed function is weakly pre-closed function,(Figure-2).

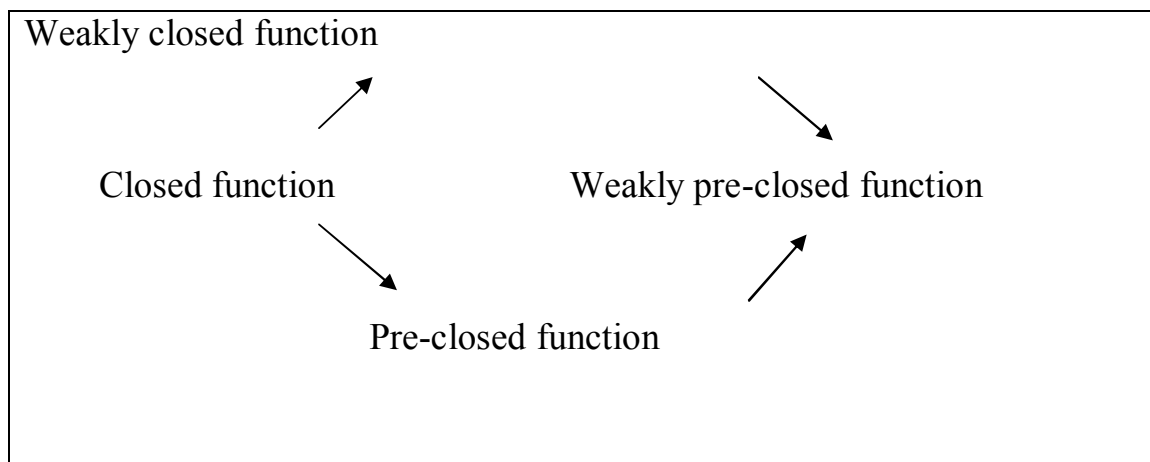


Figure2-

The proof of the following theorem is straightforward

Theorem 3.4[8]: For a function $f : X \rightarrow Y$ the following statements are equivalent

- (i) f is weakly pre-closed function
- (ii) For each regular closed subset F of X $\text{pre-Cl}(f(\text{Int}(F))) \subseteq f(\text{Cl}(\text{Int}(F)))$.
- (iii) For each pre-closed subset F of X $\text{pre-Cl}(f(\text{Int}(F))) \subseteq f(F)$.
- (iv) For each α -closed subset F of X $\text{pre-Cl}(f(\text{Int}(F))) \subseteq f(F)$.

It is very easy to prove the following lemma

Lemma 3.5: In a topological space (X, T) every closed $A \subseteq X$, if A is nowhere dense set if and only if $\text{Int}A$ nowhere dense set.

Now we will prove the main theory in this paper

Theorem 3.6: let $f : X \rightarrow Y$ be nd-preserving function from perfectly normal topological space X onto a space Y then f is WSC-function if and only if f is weakly pre-closed function

Proof. Let f be a WSC-function then $\text{S-Cl}(f(\text{Int}(F))) \subseteq f(F)$ for each closed subset F of X . let $A = \text{Int}(F)$, F closed in perfectly normal X , then F is G -delta set. hence F is nowhere dense set. By lemma 3.12 every nowhere dense set has the property Q , but f is nd-preserving function, then $f(A)$ is nowhere dense, therefore $\text{IntCl} f(A) = \text{ClInt} f(A)$, which implies that $f(F) \supseteq \text{S-Cl} f(A) = f(A) \cup \text{IntCl} f(A) = f(A) \cup \text{ClInt} f(A) = \text{P-Cl} f(A)$ and this complete the prove of this theory ■

By Claudia C. and Daniel V. T. [9] We can replace the perfect normality in upper theory by separable compact line space.

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