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Dynamics and Chaotic of Polynomials on Quasi Banach Spaces

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Abstract

In the present paper, the concepts of a quasi-metric space, quasi-Banach space have been introduced. We prove some facts which are defined on these spaces and define some polynomials on quasi-Banach spaces and studied their dynamics, such as, quasi cyclic and quasi hypercyclic. We show the existence of quasi chaotic in the sense of Devaney (quasi D-chaotic) polynomials on quasi Banach space of q-summable sequences ℓ_q , $0<q<1$ such polynomials P is defined by $P((x_i)_i)=(p(x_{i+m}))_i$ where $p:C\rightarrow C$, $p(0) = 0$. In general we also prove that P is quasi chaotic in the sense of Auslander and Yorke (quasi AY-chaotic) if and only if 0 belong to the Julia set of p, $\forall m\in N$. And then we prove that if the above polynomial P on ℓ_q , $0<q<1$ is quasi AY-chaotic then so is λP where $\lambda\in R^+$ with $\lambda\geq 1$ and P^n for each $n\geq 2$.

Keywords: Quasi-Banach space, Polynomials, Quasi Hypercyclic, Quasi Chaos, Julia set.

ديناميكية وفوضوية متعددات الحدود على فضاءات شبه بناخ

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الخلاصة

في هذا البحث تم تقديم المفاهيم الخاصة بشبه فضاء متري وشبه فضاء بناخ ومن ثم برهان بعض الحقائق المعرفة على هذه الفضاءات، حيث تم تعريف بعض متعددات الحدود المعرفة على فضاءات شبه بناخ ومن ثم دراسة ديناميكيتها مثل الشبه دوارية والشبه فوق الدوارية.لقد لاحظنا الوجود لشبه الفوضوية (نسبة الى Devaney) لمتعددات الحدود P على فضاء شبه بناخ ℓ_q عندما $0<q<1$ حيث P معرفة $P((x_i)_i):=(p(x_{i+m}))_i$، $p:C\rightarrow C$ بحيث $p(0)=0$. بصورة عامة تم اثبات ان P هي شبه فوضوية (نسبة الى Auslander و York) اذا وفقط اذا كان الصفر ينتمي الى مجموعة جوليا J(p) لكل $m\in N$. كذلك برهنا انه اذا كانت متعددة الحدود اعلاه P على ℓ_q عندما $0<q<1$ هي شبه فوضوية (نسبة الى Auslander و York) فأن λP و P^n تكون كذلك حيث $\lambda\in R^+$ ، $\lambda\geq 1$ لكل $n\geq 2$.

Introduction

Let X be a separable Banach space and $P:X\rightarrow X$ be a continuous polynomial. P is called a hypercyclic polynomial if there exist $x\in X$ such that the orbit of x under P which is denoted by $orb(P,x)=\{x,Px,P^2x,\dots,P^nx,\dots\}$ is dense in X [1]. P is cyclic polynomial if, there exist $x\in X$ such that the linear span of $orb(P,x)$ is dense in X [2]. For the polynomial $P:X\rightarrow X$, $(x_i)_i\rightarrow(p(x_{i+m}))_i$, where $X=\ell_q$

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$1\leq q<\infty$ and p:C→C is a polynomial of degree strictly greater than 1 such that p(0)=0, in [2] proved that the hypercyclicity and chaos of the polynomial P is related to the fact that 0 is fixed point belonging to the Julia set of p for each m∈N.

The sequences space ℓ_q , 0<q<1 with ${}_q\|x\|=[\sum_{i=1}^{\infty}|x|^q]^{1/q}$ is a quasi-Banach space [3] and has the separating dual [4]. Then by [5, Th.46.8] the space ℓ_q , 0<q<1 is separable.

From all the above, and since every Banach space is quasi-Banach space, but the converse is not true (see Remark 2.2) we can extend the concepts and results above such that these results are valid in any separable quasi-Banach spaces. Thus the first aim of this paper is to give a clear picture about the dynamics polynomials which are defined on quasi-Banach spaces. Also the characterization of the set of quasi cyclic vectors and the set of quasi hypercyclic vectors are studied.

The second aim is to study the chaoticity of the polynomial P: $\ell_q\to\ell_q$, 0<q<1, $P((x_i)_i):=(p(x_{i+m}))_i$ where p:C→C is a polynomial of degree greater than 1 such that p(0)=0, a necessary condition in order that P is well defined, and show that the quasi hypercyclicity and quasi chaos of this polynomial P is related to the fact that 0 belongs to the Julia set of p, and we proved some results for P.

1. Definitions and Some Results on Quasi-Metric Spaces

In this section we introduce the concepts of the quasi-metric spaces, and prove some results concerning these concepts.

Definition(1.1) [6]: Let X be a non-empty set. A function D:X×X→R is said to be a quasi-metric space if (1) $D(x,y)\geq 0$, $\forall x,y \in X$ and $D(x,y)=0$ if and only if $x=y$. (2) $D(x,y)=D(y,x)$ $\forall$ $x,y\in X$. (3) there exist constant c ≥ 1 such that $D(x,y) \leq c[D(x,z)+ D(z,y)]$ $\forall$ $x,y,z\in X$. The pair (X,D) is called a quasi-metric space.

It is clear that every metric space is a quasi-metric space but the converse may be not true (the converse is true only if c=1).

Definition(1.2): Let (X,D) be a quasi-metric space, then

(a) If $x\in X$ and ε any positive real number, then the ε-neighborhood of x denoted by $N(x,\varepsilon)$ is defined to be $\{y\in X|D(x,y) < \varepsilon\}$.

(b) The diameter of a subset A of X denoted by d(A) is defined by $\sup\{D(x,y)|\, x,y\in A\}$.

(c) A subset $\mathcal{U}$ of X is said to be open if given any point x of $\mathcal{U}$, there is a positive number ε, such that $N(x,\varepsilon) \subset \mathcal{U}$.

Proposition(1.1): Let (X,D) be a quasi-metric space, then:

(a) X and ϕ are open sets, (b) The intersection of any two open sets is an open set. (c) The union of any family of open sets is an open set.

Proof: The proof is clear by using the same manner for a metric space.

Definition(1.3): Let (X,D) be a quasi-metric space, then the set of all open subset of X form a topology on X. This topology is called the quasi-metric topology (or quasi-topology) induced on X by D.

Definition(1.4): Let (X,D) be a quasi-metric space, then

(a) A sequence $\{x_n\}$ of element of X called convergent to the point $x\in X$, if for ε > 0 there exist a positive integer N(ε) such that $D(x_n,x) < \varepsilon$, ∀n ≥ N it is denoted by ($x_n\to x$ as n→∞) , x is said to be the limit of $\{x_n\}$.

(b) A sequence $\{x_n\}$ in X called a Cauchy sequence if for ε > 0, there exist a positive integer N(ε) such that $D(x_n,x_m) < \varepsilon$, ∀n,m ≥ N.

(c) A point x is a limit point of E⊂X if every neighborhood contains a point $y \neq x$ such that $y\in E$, the set of all limit points of E is called the derived set of E and denoted by E′.

(d) E is closed set if every limit point of E is a point of E.

(e) The closure of E which is denoted by $\overline{E}$ is a closed set containing E which is formed by adding E′.

(f) The interior of E, denoted by E° is the union of all open set which are contained in E.

Definition(1.5): A quasi-metric space in which every Cauchy sequence is convergent is called a complete quasi-metric space.

The following lemmas can be proved on any quasi-normed space [3], so that it is very easy to prove it on any quasi-metric space.

Lemma(1.1): Let (X,D) be a quasi-metric space, then every convergent sequence is a Cauchy sequence.

Lemma(1.2): Any closed subspace Y of a complete quasi-metric space (X,D) is a complete quasi-metric space.

Lemma(1.3): Let E be a non-empty subset of a quasi-metric space (X,D) then

(a) $E=\overline{E}$ if and only if E is closed set.

(b) $x\in \overline{E}$ if and only if there exist a sequence $\{x_n\}$ in E such that $x_n\to x$ as $n\to\infty$.

Definition(1.6): Let (X,D) be a quasi-metric space, and let A be a subset of X, then A is said to be

(a) Quasi dense in X if $\overline{A}=X$.

(b) Quasi somewhere dense if $(\overline{A})^\circ \neq \phi$.

(c) Quasi nowhere dense if it is not quasi somewhere dense.

Definition(1.7): A quasi metric space is called separable if it contains a countable quasi dense subset.
The proof of the following proposition is very simple.

Proposition(1.2): A subset A of a quasi-topological space X is quasi dense if and only if every open subset of X contains some point of A.

Definition(1.8): Let f be a map on quasi-metric space X, f is said to be quasi transitive, if for each non-empty open subsets $\mathcal{U}$ and $\mathcal{V}$ of X, there exist $n\in N$ such that $f^n(\mathcal{U})\cap\mathcal{V} \neq \phi$.

Remark(1.1): It is clear that $f^{-n}(\mathcal{U})\cap\mathcal{V} \neq \phi$ if and only if $f^n(\mathcal{V})\cap \mathcal{U} \neq \phi$, so that we can say that the map f is quasi transitive $\Leftrightarrow$ for each non-empty open subsets $\mathcal{U}$ and $\mathcal{V}$ of X, there exist $n\in N$ such that $f^{-n}(\mathcal{U})\cap\mathcal{V} \neq \phi$.

Proposition(1.3): Let $f:X\to X$ be a continuous map on quasi-metric space (X,D), then f is a quasi transitive if and only if $\forall\ x,y\in X$ and $\forall\varepsilon > 0$, there exist $z\in X$ and $n\in N$ such that $D(x,z) < \varepsilon$ and $D(f^n(z),y) < \varepsilon$.

Proof: $\Leftarrow$ Let $\mathcal{U}$, $\mathcal{V}$ be a non-empty open subsets of X, and let $x\in\mathcal{U}$, $y\in\mathcal{V}$, then there exist $\varepsilon_1>0$ such that $N(x,\varepsilon_1) \subseteq \mathcal{U}$, there exist $\varepsilon_2 > 0$ such that $N(y,\varepsilon_2) \subseteq \mathcal{V}$. Let $\varepsilon=\min\{\varepsilon_1,\varepsilon_2\}$, thus $N(x,\varepsilon) \subseteq \mathcal{U}$ and $N(y,\varepsilon)\subseteq\mathcal{V}$, then there exist $z\in X$ and $n\in N$ such that $D(x,z) < \varepsilon$ and $D(f^n(z),y) < \varepsilon$, thus $z\in\mathcal{U}$, $f^n(z)\in\mathcal{V}$. Therefore $f^n(\mathcal{U})\cap\mathcal{V}\neq\phi$, and hence f is a quasi transitive.

$\Rightarrow$Let $x, y\in X$, $\varepsilon > 0$ and let $\mathcal{U}=N(x,\varepsilon)$, $\mathcal{V}=N(y,\varepsilon)$. Since f is a quasi transitive, there exist $n\in N$ such that $f^n(\mathcal{U})\cap\mathcal{V}\neq\phi$, thus there exist $z\in\mathcal{U}$ such that $f^n(z)\in\mathcal{V}$. Therefore $D(z,x) < \varepsilon$ and $D(f^n(z),y) < \varepsilon$.

Definition(1.9): The map f on a quasi-metric space (X,D) is said to have quasi sensitive dependence on initial conditions at $x_0\in X$, if there exist $\varepsilon > 0$ such that for any open set $\mathcal{U}\subset X$ containing x_0 there exist $y_0\in\mathcal{U}$ and $n\in N$ such that $D(f^n(x_0),f^n(y_0)) > \varepsilon$. If f has a quasi sensitive dependence on initial conditions at each $x_0\in X$, we say that f has a quasi sensitive dependence on initial conditions.

Recall that a point $x\in X$ is a periodic point for f if there exist $n\in N$ for which $f^n(x)=x$, the least such n is called its period (for $n=1$, such a point is called a fixed point).

Definition(1.10) [1],[7]**:** The map f on a metric space X is called chaotic in the sense of Auslander and York (AY-chaotic) if it is transitive and has sensitive dependence on initial conditions. While a map f is called chaotic in the sense of Devaney (D-chaotic) if it is transitive, the set of periodic points of f is dense in X and f has sensitive dependence on initial condition.

Note(1.1): We can extend chaotic map in the definition above when we define a function f on any quasi-metric space (X, D) as follows.

Definition(1.11): The map f on a quasi-metric space X is called quasi chaotic in the sense of Auslander and York (quasi AY-chaotic) if it is quasi transitive and has quasi sensitive dependence on initial conditions. While a map f is called a quasi chaotic in the sense of Devaney (quasi D-chaotic) if it is quasi transitive, the set of periodic points of f is quasi dense in X and f has quasi sensitive dependence on initial condition.

Quasi chaotic in the sense of Devaney seems to be stronger condition than quasi AY-chaos, within our frame work.

Proposition(1.4): Let (X, D) be a separable quasi-metric space then it has a countable quasi dense base.

Proof: Let $\{x_n \mid n\in N\}$ be countable quasi dense subset of X. Let $B(n,m)=N(x_n,1/m)$, where $m,n\in N$. We shall show that $\mathcal{B}=\{B(n,m)|m,n\in N\}$ is a basis for the quasi-metric topology on X. let $\mathcal{U}$ be any open subset of X and let $x\in\mathcal{U}$. Since $\mathcal{U}$ is open, there is a positive number ε such that $N(x,\varepsilon)\subset\mathcal{U}$. Choose any integer $m > (2c)/\varepsilon$, $c\geq 1$. Since $N(x,1/2m)$ is open and $\{x_n|n\in N\}$ is a quasi dense, there is some $x_n\in N(x,1/2m)$ (by Prop. 1.2), then $x\in N(x_n,1/m)$. Since $m>((2c)/\varepsilon)$, $((2c)/m)<\varepsilon$, then $N(x_n,1/m)\subset N(x,\varepsilon)$, therefore $N(x_n,1/m)\subset\mathcal{U}$.

Then $\mathcal{U}$ is the union of members of $\mathcal{B}$ (since $x\in\mathcal{U}$ and $N(x_n,1/m)\subset \mathcal{B}$). Since $\mathcal{U}$ is an arbitrary open set then $\mathcal{B}$ is a basis for the quasi-metric topology, and $\mathcal{B}$ is countable, thus X has a countable quasi dense base.

Proposition(1.5): Let A be a non-singleton subset of the quasi-metric space (X,D), then $d(A)\leq d(\overline{A})\leq cd(A)$ where $c\geq 1$ be a constant.

Proof: Let $x,y\in\overline{A}$, then $x,y\in A$ or $x,y\in A'$ or $x\in A$ and $y\in A'$. If $x,y\in A$, then it is clear that $d(\overline{A})\leq cd(A)$, where $c\geq 1$ be a constant. Now if $x,y\in A'$, then both $N(x,\varepsilon/2c_1)$ and $N(y,\varepsilon/(2c_1c_2))$ meet A, where $c_i\geq 1$, i=1,2 are constants, and ε be any positive number. Thus by choosing $a\in N(x,\varepsilon/2c_1)\cap A$ and $b\in N(y,\varepsilon/(2c_1c_2))\cap A$ we obtained that $D(x,y)\leq c_1[D(x,a)+D(a,y)]\leq c_1[D(x,a)+c_2[D(a,b)+D(b,y)]]\leq c_1[\varepsilon/2c_1+c_2d(A)+c_2\varepsilon/(2c_1c_2)] = \varepsilon/2+c_1c_2d(A)+\varepsilon/2 = c_1c_2d(A)+\varepsilon \leq cd(A)$, where $c \geq c_1c_2+\varepsilon/d(A) > 1$. Therefore $d(\overline{A})\leq cd(A)$, $c\geq 1$. Now if $x\in A$ and $y\in A'$ then $N(y,\varepsilon/c_1)$ meet A say in a where $c_1\geq 1$ be constant and ε be any positive number then $D(x,y)\leq c_1[D(x,a)+D(a,y)]\leq c_1[d(A)+\varepsilon/c_1]=c_1d(A)+\varepsilon\leq cd(A)$ where $c\geq c_1+\varepsilon/d(A)\geq 1$ therefore $d(\overline{A})\leq cd(A)$, $c\geq 1$.
And since $A\subseteq\overline{A}$, then $d(A)\leq d(\overline{A})$ and the proof is complete.

Proposition(1.6): A quasi-metric space (X,D) is complete if and only if given a countable family $\{A_n\}_{n\in N}$ of closed, non-empty subsets of X such that $A_1\supset A_2\supset\ldots\supset A_n\supset\ldots$ and $d(A_n)\to 0$ then $\bigcap_N A_n \neq \phi$.

Proof:⇒ Suppose (X,D) is a complete quasi-metric space. $\forall n\in N$ choose $a_n\in A_n$. Let $\varepsilon > 0$, and $c \geq 1$ be a constant. Then there is an integer M such that $n>M \Rightarrow d(A_n) < \varepsilon/2c$. If k and m are both integer greater than M, then $a_k,a_m\in A_{M+1}$, hence $D(a_k,a_m)\leq c[D(a_k,a_n)+D(a_n,a_m)]\leq c[d(A_{M+1})+d(A_{M+1})]\leq c((\varepsilon/2c)+(\varepsilon/2c))=\varepsilon$. Thus $\{a_n\}_{n\in N}$ is a Cauchy sequences. Since (X,D) is a complete quasi-metric space, then $a_n\to y$, then for any n, $\{a_n, a_{n+1},\ldots\}$ is also a sequence which converges to y. But A_n is closed and $\{a_n, a_{n+1},\ldots\}\subset A_n$ for each n, therefore $y\in A_n$. Since n was arbitrary, $y\in\bigcap_N A_n$, therefore $\bigcap_N A_n \neq \phi$.

$\Leftarrow$Conversely suppose that given any decreasing sequence $A_1\supset A_2\supset\ldots$ of closed, non-empty subsets of X such that $d(A_n)\to 0$, $\bigcap_N A_n \neq \phi$.

Let $\{s_n\}_{n\in N}$ be a Cauchy sequence in X. set $B_n=\{s_k | k\geq n\}$ and $A_n = \overline{B}_n$ $\forall n\in N$. Then $\{A_n\}_{n\in N}$ satisfies the above conditions, and hence $\bigcap_N A_n \neq \phi$. Choose $y\in\bigcap_N A_n$. We now show that $s_n\to y$. Let $\varepsilon>0$, then there exist an integer M such that if n > M, $d(B_n) < \varepsilon/c$, where $c\geq 1$ is a constant. By Proposition 1.6, $d(A_n)=d(\overline{B}_n)\leq cd(B_n) < c(\varepsilon/c)=\varepsilon$ then $D(s_n,y)<\varepsilon$.

Theorem(1.1): (**Quasi Bair Category Theorem**)

Let (X,D) be a non-empty complete quasi-metric space. Then the following statements hold:

(a) If X is expressed as the union of countably many subsets $A_1,A_2,\ldots,A_n,\ldots$. Then at least one of the A_n is a quasi somewhere dense. That is, for one of the A_n, $\overline{A}_n$ contains an open subsets of X.

(b) if $\mathcal{U}_1$, $\mathcal{U}_2$,… are countably many quasi dense open subsets of X, then $\bigcap_N \mathcal{U}_n$ is quasi dense in X, that is $cl(\bigcap_N \mathcal{U}_n)=X$.

Proof: (a) If (a) is false, then there exist countable family $\{A_n\}_{n\in N}$ of subsets of X such that $X=\bigcup_N A_n$ but $(\overline{A})^\circ=\phi$ for each $n\in N$ then $\overline{A}_n\neq X$ $\forall n$. Select $b_1\in X-\overline{A}_1$, since $X-\overline{A}_1$ is open, there exist a positive number $\varepsilon_1<1$ such that $N(b_1,\varepsilon_1)\subset X-\overline{A}_1$. Set $B_1=N(b_1,\varepsilon_1/2)$ (see Figure-1). Then $\overline{B}_1=N(b_1,\varepsilon_1)$, hence $\overline{B}_1\bigcap\overline{A}_1=\phi$. Now B_1 is a non-empty open subset of X and therefore $B_1\not\subset\overline{A}_2$. Choose $b_2\in B_1-\overline{A}_2$. Since $B_1-\overline{A}_2$ is open there exist $\varepsilon_2>0$ such that $N(b_2,\varepsilon_2)\subset B_1-\overline{A}_2$. With no loose of generality, we can take $\varepsilon_2<(1/2)$ in further requiring. Set $B_2= N(b_2,\varepsilon_2/2)$, then $B_2\subset B_1$ and $\overline{B}_2\cap\overline{A}_2=\phi$. By using the same manner above we can obtain a decreasing sequence of open ε_n-neighborhoods $B_1\supset B_2\supset\ldots B_n\supset\ldots$ such that $\overline{B}_n\bigcap\overline{A}_n=\phi$ and $\varepsilon_n<(1/n)$. Then $\overline{B}_1\supset\overline{B}_2\supset\ldots\supset\overline{B}_n\supset\ldots$ and $d(B_n)\to 0$. Then (by Prop. 1.6) $\bigcap_N\overline{B}_n\neq\phi$. Pick $x\in\bigcap_N B_n$. Then $x\in A_n$ for some n, since $\bigcup_N A_n = X$. But then $x\in\overline{A}_n\bigcap\overline{B}_n$ which is impossible, since $\overline{A}_n$ and $\overline{B}_n$ are disjoint. Therefore (a) is proved.

(b) Suppose $\{\mathcal{U}_n\}_{n\in N}$ is countable family of quasi dense open subsets of X. In order to prove that $\bigcap_N \mathcal{U}_n$ is quasi dense, it sufficient to prove that each neighborhood of any point of X meet $\bigcap_N \mathcal{U}_n$. Choose any $x\in X$ and any $\varepsilon > 0$, we will show that $N(x,\varepsilon)\cap(\bigcap_N \mathcal{U}_n)\neq\phi$ (this suffices to prove (b) since the collection of ε-neighborhoods is a basis for topology induced by D). Set $T=\overline{N(x,\varepsilon/2)}$, then $T\subset N(x,\varepsilon)$, we now show that $T\cap(\bigcap_N \mathcal{U}_n)\neq\phi$. Since T is closed, the subspace T is itself a complete quasi-metric space (Lemma 1.2). Set $A_n=T-\mathcal{U}_n$. Since $A_n=T-\mathcal{U}_n=T\cap(X-\mathcal{U}_n)$, the intersection of two closed subsets of X, A_n is closed in both X and T. Suppose A_n is quasi somewhere dense. Then there exist $t\in T$ and $\delta>0$ such that $N(t,\delta)\cap T\subset\overline{A}_n\cap T=A_n$. Therefore $N(t,\delta)\cap(T-A_n)=\phi$. Now $t\in T=\overline{N(x,\varepsilon/2)}$ (see figure-2), hence $N(t,\delta)$ meets $N(x,\varepsilon/2)$ in some point z. We may choose $\delta'>0$ such that $N(z,\delta')\subset N(t,\delta)\cap N(x,\varepsilon/2)$. But since $\mathcal{U}_n$ is quasi dense, $N(z,\delta')$ intersect $\mathcal{U}_n$ say in z'. Then $z'\in T\cap N(t,\delta)\subset A_n$. But $A_n=T-\mathcal{U}_n$ and hence $z'\in T-\mathcal{U}_n$, that is $\delta'\notin\mathcal{U}_n$, a contradiction. Therefore A_n must be quasi nowhere dense in T. Then by (a), $T \neq \bigcup_{N} A_n$ (note that T is a complete quasi-metric space), thus there exist $y \in T - \bigcup_{N} A_n$. Therefore since $A_n = T - \mathcal{U}_n$, $y \in T \cap (\bigcap_{N} \mathcal{U}_n)$. Then $T \cap (\bigcap_{N} \mathcal{U}_n) \neq \phi$, and hence $N(x,\varepsilon) \cap (\bigcap_{N} \mathcal{U}_n) \neq \phi$. This completes the proof of (b).

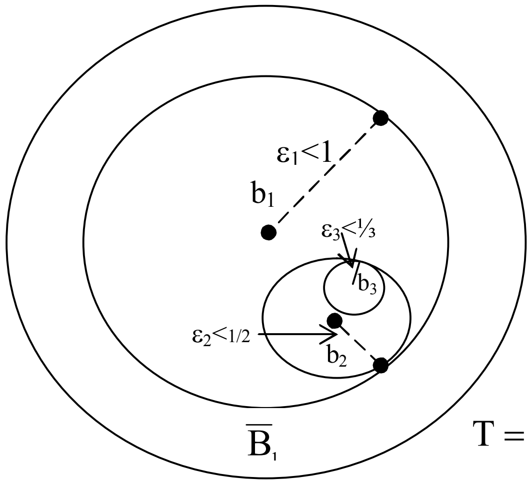

Figure 1-

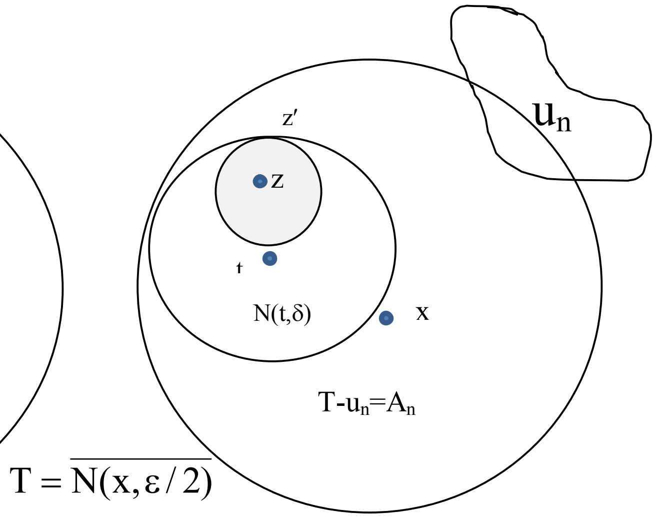

Figure 2-

2. Polynomial on Quasi-Banach Spaces

In this section we will give the definition of quasi-Banach spaces and then study the dynamics (quasi transitive, quasi chaotic, quasi hypercyclic and quasi cyclic) of polynomials which are defined on quasi-Banach spaces.

Definition(2.1) [6]: If X is a vector space over a field F. A quasi-norm on X is a function ${}_q\|.\|: X \to R$ satisfies the following axioms: (1) ${}_q\|x\| \geq 0$ $\forall x \in X$, ${}_q\|x\|=0$ if and only if $x=0$. (2) ${}_q\|\lambda x\| = |\lambda|\, {}_q\|x\|$, $\forall x$, $\forall \lambda \in F$. (3) there exist a constant c ≥ 1 such that ${}_q\|x+y\| \leq c[{}_q\|x\| + {}_q\|y\|]$, $\forall x,y \in X$. The pair (X, ${}_q\|x\|$) is called a quasi-normed space. We say simply that X is a quasi-normed space.

Remark(2.1): It is easy to see that every quasi-normed space is a quasi-metric space (by defined $D(x,y) = {}_q\|x-y\|$) [6], also every normed space is a quasi-normed space, but the converse, in general, may not be true (the converse is true only if c=1) [6].

Note(2.1): We will use in this section all definitions that used in the proceeding section, but on the quasi-normed space. Thus we will only replaced the quasi-metric (X, D) by quasi-normed (X, ${}_q\|.\|$).

Remark(2.2): Every Banach space is a quasi-Banach space, but the converse may be not true [6].

Examples (2.1):

(1) The sequences space $\ell_p = \left\{ \{x_i\}, x_i \in R \text{ or } C, \text{ such that } \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$, $0 < p < 1$, with quasi norm ${}_q\|x\| = \left[\sum_{i=1}^{\infty} |x_i|^p \right]^{1/p}$ is a quasi-Banach space, but it is not Banach space [3].

(2) The space of measurable functions L_p, $0 < p < 1$ is a quasi-Banach space, but it is not Banach space [6, Th. 2.1].

Definition(2.2)[8]: Let X, Y be vector spaces. A map f :X→Y is a continuous m-homogenous polynomial (or homogenous polynomial of degree m) if there exists a continuous m-linear mapping $\overline{f}$:X^m→Y such that $f(x)=\overline{f}(x,\ldots,x)$ ∀x∈X. We say that $\overline{f}$ is associated with f or that $\overline{f}$ generates f. If P:X→Y is a finite sum $P=\sum_{k=0}^{m} p_k$ of K-homogenous polynomials p_k:X→Y, then P is called a (continuous) polynomial (of degree at most m). Note that every operator (bounded linear transformation) is a 1-homogenous polynomial.

Note(2.1):

(1). We denote by G_δ-set the countable intersection of open sets [9].

(2). In the following sections X will always denote a separable quasi Banach space.

2.1 Quasi Cyclic Polynomials

Definition (2.3): Let P:X→X be a continuous polynomial on the quasi- Banach space X. $\mathscr{P}$ is said to be a quasi cyclic polynomial if there exists x∈X such that the linear span of orb(P,x) is a quasi dense in X, or equivalently {q(P)x: q is a polynomial} is quasi dense in X. Such a vector x is said to be a quasi cyclic vector for P.

Proposition(2.1): Let P:X→X be a continuous polynomial on the quasi- Banach space X. let $\mathscr{P}$ be the set of all polynomials in P. if $\{\mathcal{U}_n\}_{n=1}^{\infty}$ is a basis for the quasi topology on X, then $\bigcap_n[\bigcup_{A\in\wp} A^{-1}(\mathcal{U}_n)]$ is the set of all quasi cyclic vectors for P. Hence the set of all quasi cyclic vectors for P is G_δ-set.

Proof: Note that x is a quasi cyclic vector for P if and only if $\overline{\{q(P): q \text{ is polynomial}\}}$=X if and only if∀n≥0 there exist A in $\mathscr{P}$ such that Ax∈$\mathcal{U}_n$ if and only if x∈$A^{-1}(\mathcal{U}_n)$ if and only if $x\in\bigcup_{A\in\wp} A^{-1}(\mathcal{U}_n)$ if and only if $x\in\bigcap_n[\bigcup_{A\in\wp} A^{-1}(\mathcal{U}_n)]$. Since $\mathcal{U}_n$ an open set ∀n≥0 and A∈$\mathscr{P}$ is continuous then $A^{-1}(\mathcal{U}_n)$ is an open set ∀n≥0 thus $\bigcup_{A\in\wp} A^{-1}(\mathcal{U}_n)$ is open. Therefore $\bigcap_n[\bigcup_{A\in\wp} A^{-1}(\mathcal{U}_n)]$ is countable intersection of open set, then it is G_δ-set.

Theorem(2.1): Let P:X→X be a continuous polynomial on the quasi- Banach space X, then the following statements are equivalent:

(1) P has a quasi dense set of quasi cyclic vectors.

(2) For each non empty open subsets $\mathcal{W}$ and $\mathcal{V}$ of X, there exist polynomial q in P such that q(P)$\mathcal{W}\cap\mathcal{V}\neq\phi$.

(3) For each x,y∈X, there exist sequences $\{x_k\}_{k\in N}$ in X, $\{q_k\}_{k\in N}$ of polynomials in P such that $x_k\to x$ and $q_k(P)x_k\to y$ (or $_q\|x_k-x\|\to 0$ and $_q\|q_k(P)x_k-y\|\to 0$ as k→∞).

(4) For each x,y∈X and each neighborhood $\mathcal{V}$ of zero in X, there exist z∈X and a polynomial q in P such that $z-x\in\mathcal{V}$ and q(P)$z-y\in\mathcal{V}$.

Proof. (1)→(2), let $\mathcal{W}$, $\mathcal{V}$ be non empty open subsets of X. Let $\mathscr{P}$ be the set of all polynomials in P, let $\{\mathcal{U}_n\}_{n\in N}$ be basis for the quasi topology on X. Since $\bigcap_n[\bigcup_{A\in\wp} A^{-1}(\mathcal{U}_n)]$ is the set of all quasi cyclic vectors for P (from Prop. 2.1), then $\bigcap_n[\bigcup_{A\in\wp} A^{-1}(\mathcal{U}_n)]$ is quasi dense in X (by(1)), thus $\bigcup_{A\in\wp} A^{-1}(\mathcal{U}_n)$ is quasi dense in X for all n∈N. Now, assume that ∀A∈$\mathscr{P}$, A($\mathcal{W}$)∩$\mathcal{V}$=ϕ, then $\mathcal{W}\cap A^{-1}(\mathcal{V})$=ϕ but $\mathcal{V}=\cup\mathcal{U}_m$,

$\mathcal{U}_m \in \{\mathcal{U}_n\}_{n\in N}$ therefore $\mathcal{W} \cap A^{-1}(\mathcal{U}_m)=\phi$, $\mathcal{W} \cap [\bigcup_{A\in\wp} A^{-1}(\mathcal{U}_m)]=\phi$ which is a contradiction with the quasi density of $\bigcup_{A\in\wp} A^{-1}(\mathcal{U}_m)$.

(2)→(3), let $x,y\in X$, let $B_1=N(x,1)$ and $B'_1=N(y,1)$ by (2) there exist a polynomial q_1 in P such that $q_1(P)B_1\cap B'_1\neq\phi$, then there exist $x_1\in B_1$ such that $q_1(P)x_1\in B'_1$. Now, let $B_2=N(x,(1/2))$, $B'_2=N(y,(1/2))$ by (2) there exist a polynomial q_2 in P such that $q_2(P)B_2\cap B'_2\neq\phi$, thus there exist $x_2\in B_2$ such that $q_2(P)x_2\in B'_2$,…, and so on. Therefore we get sequences $\{x_k\}$ in X, $x_k\in B_k$, $\forall k\geq 1$ and $\{q_k\}$ of polynomials in P such that $q_k(P)x_k\in B'_k$ $\forall k\geq 1$. Then $_q\|x_k-x\|<1/k$ and $_q\|q_k(P)x_k-y\|<1/k$, thus we get $x_k\to x$ and $q_k(P)x_k\to y$ as $k\to\infty$.

(3)→(1), let $\wp$ be the set of all polynomials in P, let $\{\mathcal{U}_n\}_{n\in N}$ be basis for the quasi topology on X. We want to prove that $\bigcap_n [\bigcup_{A\in\wp} A^{-1}(\mathcal{U}_n)]$ is quasi dense in X. for a fixed n, let $y\in X$ and $x_n\in\mathcal{U}_n$, by (3) there exist sequences $\{x_k\}$ in X and $\{q_n\}$ of polynomial in P such that $x_k\to y$ and $q_k(P)x_k\to x_n$, therefore for all large k, $q_k(P)x_k\in\mathcal{U}_n$, $x_k\in A^{-1}(\mathcal{U}_n)$, $A=q_k(P)\in\wp$ then $x_k\in \bigcup_{A\in\wp} A^{-1}(\mathcal{U}_n)$ for all large k. Therefore there exist subsequence $\{x'_k\}$ of sequence $\{x_k\}$ such that $x'_k\in \bigcup_{A\in\wp} A^{-1}(\mathcal{U}_n)$ and $x'_k\to y$. Then $\bigcup_{A\in\wp} A^{-1}(\mathcal{U}_n)$ is quasi dense in X, and since A is continuous, thus $A^{-1}(\mathcal{U}_n)$ is open, then $\bigcap_n [\bigcup_{A\in\wp} A^{-1}(\mathcal{U}_n)]$ is quasi dense in X (from Th. 1.1). Hence the set of all quasi cyclic vector for P is quasi dense in X (from Prop. 2.1).

(3)→(4), let $x,y\in X$ and $\mathcal{V}$ be neighborhood of zero in X, by (3) there exist sequence $\{x_k\}_{k\in N}$ in X, $\{q_k\}_{k\in N}$ of polynomials in P such that $x_k\to x$ and $q_k(P)x_k\to y$. Let $\varepsilon>0$, then there exist $\ell>0$ such that $_q\|x_k-x\|<\varepsilon$ and $_q\|q_k(P)x_k-y\|<\varepsilon$ $\forall k>\ell$. Thus, since $\mathcal{V}$ is a neighborhood of zero there exist $k\in N$ such that $x_k-x\in\mathcal{V}$ and $q_k(P)z-y\in\mathcal{V}$. Hence we get $z-x\in\mathcal{V}$ and $q_k(P)z-y\in\mathcal{V}$ by taking $z=x_k$.

(4)→(3), let $x,y\in X$, let $B_1=N(0,1)$ by (4) there exist $x_1\in X$, a polynomial q_1 in P such that $x_1-x\in B_1$ and $q_1(P)x_1-y\in B_1$. Let $B_2=N(0,(1/2))$, by (4) there exist $x_2\in X$, a polynomial q_2 in P such that $x_2-x\in B_2$ and $q_2(P)x_2-y\in B_2$,…, and so on. Then we get sequences $\{x_k\}$ in X, $\{q_k\}$ of polynomials in P such that $x_k-x\in B_k$ and $q_k(P)x_k-y\in B_k$ $\forall k$. Then $_q\|x_k-x\|<1/k$ and $_q\|q_k(P)x_k-y\|<1/k$, and hence $x_k\to x$ and $q_k(P)x_k\to y$ as $k\to\infty$.

2.2 Quasi Hypercyclic Polynomial

Definition(2.5): Let X be a quasi-Banach space and $P:X\to X$ be a continuous polynomial on X. P is said to be quasi hypercyclic if there exists $x\in X$ such that $orb(P,x)=\{P^n x: n\geq 0\}$ is a quasi dense in X, and such a vector x is called a quasi hypercyclic vector for P.

Note(2.3): It is clear that every quasi hypercyclic polynomial is a quasi cyclic polynomial. Also every quasi hypercyclic polynomial has quasi dense range.

Proposition(2.3): Let $P:X\to X$ be a continuous polynomial on the quasi- Banach space X. If $\{\mathcal{U}_i\}_{i\in N}$ is a basis for the quasi topology on X, then $\bigcap_i [\bigcup_n P^{-n}(\mathcal{U}_i)]$ is the set of all quasi hypercyclic vectors for P. Hence the set of all quasi hypercyclic vectors for P is G_δ-set.

Proof: The vector x is a quasi hypercyclic vector for Pif and only if $\overline{orb(P,x)}=X$if and only if $\forall i$ there exist $n\in N$ such that $P^n x\in\mathcal{U}_i$ if and only if$x\in P^{-n}(\mathcal{U}_i)$if and only if$x\in \bigcup_{n\in N} P^{-n}(\mathcal{U}_i)$if and only if$x\in \bigcap_{i\in N}[\bigcup_{n\in N} P^{-n}(\mathcal{U}_i)]$. Therefore $\{x: \overline{\{orb(P,x)\}}=X\}=\bigcap_{i\in N}[\bigcup_{n\in N} P^{-n}(\mathcal{U}_i)]$. Now, since $\mathcal{U}_i$ is an open set $\forall i\in N$ and P is continuous then $P^{-n}(\mathcal{U}_i)$ is an open set $\forall n\geq 0$ thus $\bigcup_{n\in N} P^{-n}(\mathcal{U}_i)$ is open $\forall i\in N$ then $\bigcap_{i\in N}[\bigcup_{n\in N} P^{-n}(\mathcal{U}_i)]$ is a countable intersection of open set, then it is a G_δ-set.

Theorem(2.2): Let X be a quasi Banach space with no isolated point and let P:X→X be a continuous polynomial on X. Then the following statements are equivalent:

(1) P is quasi hypercyclic,

(2) P is quasi transitive,

(3) For each $x,y\in X$, there exist sequences $\{x_k\}_{k\in N}$ in X, $\{n_k\}$ in N such that $x_k\to x$ and $P^{n_k}x_k\to y$.

(4) For each $x,y\in X$, each neighborhood $\mathcal{W}$ of zero in X there are $z\in X$, $n\in N$ such that $x-z\in\mathcal{W}$ and $P^n z-y\in\mathcal{W}$.

Proof: (1)→(2), Let $\mathcal{U}$, $\mathcal{V}$ be non-empty open subsets of X. since P is quasi hypercyclic then there exist n,k∈N, n≥k such that $P^n x\in\mathcal{U}$ and $P^k x\in\mathcal{V}$. To prove that $P^{n-k}(\mathcal{V})\cap\mathcal{U}\neq\phi$, suppose $P^{n-k}(\mathcal{V})\cap\mathcal{U}=\phi$, then $\forall y\in\mathcal{V}$, $P^{n-k}y\notin\mathcal{U}$ this a contradiction by taking $y=P^k x$.

(2)→(3), Let $x,y\in X$, let $B_1=N(x,1)$ and $B'_1=N(y,1)$ by (2) there exist $n_1\in N$ such that $P^{n_1}(B_1)\cap B'_1\neq\phi$ then there exist $x_1\in B_1$ such that $P^{n_1}x_1\in B'_1$. Now let $B_2=N(x,1/2)$, $B'_2=N(y,1/2)$ by (2) there exist $n_2\in N$ such that $P^{n_2}(B_2)\cap B'_2\neq\phi$ then there exist $x_2\in B_2$ such that $P^{n_2}x_2\in B'_2$,…, and so on. Therefore we get sequences $\{x_k\}$ in X, $x_k\in B_k$ $\forall k\geq 1$ and $\{n_k\}$ in N such that $P^{n_k}x_k\in B'_k\,\forall k\geq 1$. Hence $_q\|x_k-x\|<1/k$ and $_q\|P^{n_k}x_k-y\|<1/k$, then we get $x_k\to x$ and $P^{n_k}x_k\to y$ as $k\to\infty$.

(3)→(1), Let $\{\mathcal{U}_n\}_{n\in N}$ be a basis for the quasi topology on X. For a fixed n, let $y\in X$ and $x_n\in\mathcal{U}_n$, by (3) there exist sequences $\{x_k\}$ in X and $\{n_k\}$ in N such that $x_k\to y$ and $P^{n_k}x_k\to x_n$, therefore for all large k, $P^{n_k}x_k\in\mathcal{U}_n$, $x_k\in P^{-n_k}(\mathcal{U}_n)$, then $x_k\in\bigcup_k P^{-n_k}(\mathcal{U}_n)$ for all large k. Therefore there exist subsequence $\{x'_k\}$ of $\{x_k\}$ such that $x'_k\in\bigcup_k P^{-n_k}(\mathcal{U}_n)$ and $x'_k\to y$. Then $\bigcup_k P^{-n_k}(\mathcal{U}_n)$ is quasi dense in X, and since P is continuous, thus $P^{-n_k}(\mathcal{U}_n)$ is open, $\bigcup_k P^{-n_k}(\mathcal{U}_n)$ is open, then $\bigcap_n[\bigcup_k P^{-n_k}(\mathcal{U}_n)]$ is quasi dense in X (from Th. 1.1). But (by Prop. 2.3) $\bigcap_n[\bigcup_k P^{-n_k}(\mathcal{U}_n)]$=the set of all quasi hypercyclic vector for P, then P is quasi hypercyclic polynomial.

(3)→(4), Let $x,y\in X$ and $\mathcal{W}$ be a neighborhood of zero in X, by (3) there exist sequences $\{x_k\}$ in X, $\{n_k\}$ in N such that $x_k\to x$ and $P^{n_k}x_k\to y$, thus for ε>0 there exist m∈N such that $_q\|x_k-x\|<\varepsilon$ and $_q\|P^{n_k}x_k-y\|<\varepsilon$, $\forall k>m$. Then because $\mathcal{W}$ is a neighborhood of zero, there exist k∈N such that $x_k-x\in\mathcal{W}$ and $P^{n_k}x_k-y\in\mathcal{W}$. Therefore by taking $z=x_k$, $n_k=n$ we get $z\in X$, $z-x\in\mathcal{W}$ and $P^{n_k}z-y\in\mathcal{W}$.

(4)→(3) Let $x,y\in X$, let $B_1=N(0,1)$ by (4) there exist $z_1\in X$, $n_1\in N$ such that $z_1-x\in B_1$ and $P^{n_1}z_1-y\in B_1$. Let $B_2=N(0,1/2)$, by (4) there exist $z_2\in X$, $n_1\in N$ such that $z_2-x\in B_2$ and $P^{n_2}z_2-y\in B_2$,…, and so on. Then we get sequences $\{z_k\}$ in X, and $\{n_k\}$ in N such that $z_k-x\in B_k$ and $P^{n_k}z_k-y\in B_k$ $\forall k$. Then $_q\|z_k-x\|<1/k$ and $_q\|P^{n_k}x_k-y\|<1/k$, thus $z_k\to x$ and $P^{n_k}z_k\to y$ as $k\to\infty$. Therefore by taking $z_k=x_k$ we get sequences $\{x_k\}$ in X, $\{n_k\}$ in N such that $x_k\to x$ and $P^{n_k}x_k\to y$.

Remark(2.3): From Remark 1.1, Proposition 2.3 and the above theorem we have the following: If P is a quasi hypercyclic polynomial, then the set of all quasi hypercyclic vectors for P is a quasi dense G_δ-set, since if P is a quasi hypercyclic then for every non-empty open subsets $\mathcal{U}$ and $\mathcal{V}$ of X, there is a non-negative integer n such that $P^{-n}(\mathcal{U})\cap\mathcal{V}\neq\phi$, thus $\bigcup_{n\in N}P^{-n}(\mathcal{U}_i)$ intersect any open subsets of X and then it is a quasi dense set in X. Therefore by (Th. 1.1) $\{x:\overline{orb(P,x)}=X\}$ is quasi dense.

Proposition(2.4): Let X be quasi-Banach space and P:X→X be a continuous polynomial on X, and let $x\in X$ be a quasi hypercyclic vector for P, then $\inf\{_q\|P^n x\|:n\geq 0\}=0$ and $\sup\{_q\|P^n x\|:n\geq 0\}=\infty$.

Proof: Suppose that $\inf\{_q\|P^n x\|:n\geq 0\}=t>0$ since 0∈X and x is a quasi hypercyclic vector for P then there exist a sequence $\{P^{n_j}x\}$ such that $P^{n_j}x\to 0$, thus there exist k∈N such that $_q\|P^{n_j}x\|<t$ $\forall j<k$ which is a contradiction with $\inf\{_q\|P^n x\|:n\geq 0\}=t>0$. Therefore $\inf\{_q\|P^n x\|:n\geq 0\}=0$. Now suppose that $\sup\{_q\|P^n x\|:n\geq 0\}=k<\infty$. Let $y\in X$ such that $_q\|y\|>k$. Since x is a quasi hypercyclic vector for P, there

exist a sequence$\{P^{n_j}x\}$ in orb(P,x) such that $P^{n_j}x \to y$, thus $_q\|P^{n_j}x\| \to {}_q\|y\|$ (see, [10], lemma 2.2.4), but $\sup\{_q\|P^nx\|:n\geq 0\}=k$, then $_q\|P^nx\|\leq k$ $\forall k$. Therefore $_q\|y\|\leq k$ a contradiction.
Now to prove the main theorems we need the following:

Theorem(2.3)[11]: Suppose that $f_n(z)$ is analytic in the region Ω, and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in region Ω, uniformly on every compact subset of Ω. Then $f(z)$ is analytic in Ω. Moreover $f'_n(z)$ converges uniformly to $f'(z)$ on every compact subsets of Ω.

Definition(2.6) [11]: A family of analytic functions $\{f_n\}$ defined on an open subsets D of $\overline{C}=C\cup\{\infty\}$ is called a normal family if every infinite subsets of $\{f_n\}$ contains a subsequence which converges uniformly on every compact subsets of D. The family $\{f_n\}$ is not normal at z_0 if the family fails to be normal in every neighborhood of z_0.

Definition(2.7)[12]: The Julia set for an analytic function f on $\overline{C}$ denoted by J(f) is the set $\{z\in\overline{C} \mid \{f^n\}_{n\in N}$ is not normal on any neighborhood of $z\}$.

Observe that an important role in iteration theory is played by the periodic points. A periodic point z of period k of an analytic function f is called a repelling periodic point if $|(f^k)'(z)|>1$ [7].

Theorem(2.4)[1]: Let f be an entire function, then the Julia set of f is the closure of the set of repelling periodic points of f.

Theorem(2.5) [12]: Let $f:\overline{C}\to\overline{C}$ be an entire function then $J(f)=J(f^n)$ $\forall n\geq 2$.

Proposition(2.8)[2]: Let f:C→C be an entire function. If $0\in J(f)$ then $0\in J(\lambda f)$ where $\lambda\in R^+$ such that $\lambda\geq 1$.

Theorem(2.6)[1]: Let P:C→C be a polynomial with degree deg(P)≥2. Given an element $x_0\in C$ in the Julia set of P, a neighborhood $\mathcal{U}\subset C$ of x_0, $\varepsilon>0$ and a finite collection $\{z_1,\ldots,z_n\}\subset C$, then there are $x_i\in\mathcal{U}$, i=1,…,n and m∈N such that $|P^m(x_i)-z_i|<\varepsilon$, i=1,…,n.

Theorem(2.7): Let $X=\ell_q$, 0<q<1 and m∈N, P:X→X be a continuous polynomial given by $P((x_i)_i):=(p(x_{i+m}))_i$ where p:C→C is a polynomial of degree strictly greater than 1 such that p(0)=0. Then the following conditions are equivalent:
1) P is quasi AY-chaotic,
2) P is quasi hypercyclic,
3) P has quasi sensitive dependence on initial condition,
4) 0 belongs to the Julia set of p.

Proof: (1)→(2) since P is quasi AY-chaotic then P is quasi transitive, therefore P is quasi hypercyclic (by Th. 2.2).
(2)→(3): Let ε=1, $x\in X$, δ>0 since the set $\{x\in X|x=(x_1,\ldots,x_k,0,0,\ldots),k\in N\}$ is quasi dense in ℓ_q, then there exist k∈N such that $_q\|x-(x_1,\ldots,x_k,0,0,\ldots)\|<\delta$. Since P is quasi hypercyclic, then there exist $z\in X$ quasi hypercyclic vector for P such that $_q\|x-z\|<\delta$ (by Remark 2.3). Now, since $\overline{x}:=(x_1,\ldots,x_k,0,0,\ldots)$ then $P^k\overline{x}=0$, thus $P^n\overline{x}=0$ $\forall n\geq k$. Then $_q\|P^n\overline{x}\|=0$ $\forall n\geq k$. Now, since z is quasi hypercyclic vector then (by Prop. 2.4), $\sup\{_q\|P^nz\|:n\geq 0\}=\infty$, thus we get n∈N satisfying $_q\|P^nz\|>2^{\frac{2}{q}+1}$, 0<q<1, then $2^{\frac{2}{q}+1}<{}_q\|P^nz\|\leq c[_q\|P^nz-P^n\overline{x}\|+{}_q\|P^n\overline{x}\|]$ but $c=2^{\frac{1}{q}}$ (see [3]), then $2^{\frac{2}{q}+1}<{}_q\|P^nz\|\leq 2^{\frac{1}{q}}[_q\|P^nz-P^n\overline{x}\|+{}_q\|P^n\overline{x}\|]$, then $2^{\frac{1}{q}+1}<{}_q\|P^nz-P^n\overline{x}\|\leq 2^{\frac{1}{q}}[_q\|P^nx-P^nz\|+{}_q\|P^n\overline{x}-P^nx\|]$, thus $2<{}_q\|P^nx-P^nz\|+{}_q\|P^n\overline{x}-P^nx\|$, and since ε=1 then $_q\|P^nx-P^nz\|>\varepsilon$ or $_q\|P^n\overline{x}-P^nx\|>\varepsilon$. Therefore P has quasi sensitive dependence on initial conditions.
(3)→(4): suppose $0\notin J(p)$, then $\{p^n\}_{n\in N}$ is normal on some neighborhood $\mathcal{U}$ of 0 on which a subsequences of $\{p^n\}$ converges uniformly to a finite analytic function (it can not converge to ∞ since

$p^n(0)=0\ \forall n$), then (by Th. 2.3) the derivatives also converge, therefore $\{(p^n)'\}_{n\in N}$ is normal on some neighborhood of 0, thus $\{(p^n)'\}_{n\in N}$ is uniformly bounded on some neighborhood of 0, that is there existδ, M>0 such that $|(p^n)'(z)|\le M\ \forall n\in N, \forall z\in C, |z|<\delta$ then $|p^n z|=|\int(p^n)'(z)dz|\le\int_0^z (p^n)'(z)dz\le\max|(p^n)'(z)|\int_0^z|dz|\le M|z|$. Then $|p^n z|\le M|z|, \forall n\in N, \forall z\in C, |z|<\delta$. Since $P((x_i)_i):=(p(x_{i+m}))_i$ then

$P(x_1,x_2,\ldots)=(p(x_{1+m}),p(x_{2+m}),\ldots)$,
$P^2(x_1,x_2,\ldots)=(p^2(x_{1+2m}),p^2(x_{2+2m}),\ldots)$,
$\vdots$
$P^n(x_1,x_2,\ldots)=(p^n(x_{1+nm}),p^n(x_{2+nm}),\ldots)$.

Thus ${}_q\|P^n x\|=[\sum_{i=1}^{\infty}|p^n(x_{i+nm})|]^{1/q}\le[\sum_{i=1}^{\infty}|p^n(x_i)|]^{1/q}\le M[\sum_{i=1}^{\infty}|x_i|^q]^{1/q}=M_q\|x\|\ \forall n\in N$ and $\forall x\in X$, ${}_q\|x\|<\delta$. Thus we have $\forall\varepsilon>0$ there exist$\delta=(\varepsilon/M)$ such that $\forall n\in N$, ${}_q\|x\|<\delta$ but ${}_q\|P^n x-P^n 0\|={}_q\|P^n x\|<M_q\|x\|<M\delta=\varepsilon$. Therefore P does not have quasi sensitive dependence on initial conditions at 0 which is a contradiction with assumption. Hence $0\in J(p)$.

(4)→(1), Since we already know that (2) implies (3) and that (2)+(3)=1, we just need to prove that (4) implies (2). By Proposition 1.3 to show that P is quasi transitive is to show that $\forall x,y\in X$, $\varepsilon>0$ there exist $n\in N$ and $\bar{y}\in X$ such that ${}_q\|\bar{y}-y\|<\varepsilon$, ${}_q\|P^n\bar{y}-x\|<\varepsilon$. Let $x,y\in X$, $\varepsilon>0$. Let $k\in N$ such that ${}_q\|x-(x_1,\ldots,x_k,0,0,\ldots)\|<\varepsilon/2^{(q+1)/q}$, ${}_q\|y-(y_1,\ldots,y_k,0,0,\ldots)\|<\varepsilon/2^{(q+1)/q}$. Since $0\in J(p)$, $\{x_1,\ldots,x_k\}\subset C$, pick $\delta:=\varepsilon/(2^{(q+1)/q}k^{1/q})>0$, then (by Th. 2.6), we find n>K, $\{z_1,\ldots,z_k\}\subset C$ with $|z_i|<\delta$ and $|p^n z_i-x_i|<\delta$, $i=1,2,\ldots,k$. Define, then $\bar{y}:=(y_1,\ldots,y_k,0,\ldots,0,z_1^{nm+1},\ldots,z_k,0,\ldots)$ (where z_1^{nm+1} means that z_1 is in (nm+1) position), thus $\bar{y}\in X$ and $\|\bar{y}-y\|\le c[{}_q\|\bar{y}-(y_1,\ldots,y_k,0,\ldots)\|+{}_q\|y-(y_1,\ldots,y_k,0,\ldots)\|]< c\left[\sum_{i=1}^{k}|z_i|^q\right]^{1/q}+\varepsilon/2^{(q+1)/q}<2^{(1/q)}k^{(1/q)}\delta+(2^{(1/q)}\varepsilon/2^{(q+1)/q})=2^{(1/q)}k^{(1/q)}\varepsilon/(2^{(q+1)/q}k^{(1/q)})+2^{(1/q)}\varepsilon/2^{(q+1)/q}=(\varepsilon/2)+(\varepsilon/2)=\varepsilon$. Now, $P^n\bar{y}=(p^n z_1,\ldots,p^n z_k,0,0,\ldots)$, then ${}_q\|P^n\bar{y}-x\|\le c[{}_q\|P^n\bar{y}-(x_1,\ldots,x_k,0,\ldots)\|+{}_q\|x-(x_1,\ldots,x_k,0,\ldots)\|]\le c_q\|(p^n z_1-x_1,\ldots,p^n z_k-x_k,0,\ldots)\|+(c\varepsilon/2^{(q+1)/q})=2^{1/q}\left[\sum_{i=1}^{k}|p^n z_i-x_i|^q\right]^{1/q}+(2^{(1/q)}\varepsilon/2^{(q+1)/q})<2^{(1/q)}k^{(1/q)}\varepsilon/(2^{(q+1)/q}.k^{(1/q)})+(\varepsilon/2)=\varepsilon$.

Therefore P is quasi transitive, then (by Th.2.2) P is quasi hypercyclic.

Theorem(2.8): Let $X=\ell_q$, $0<q<1$, $m\in N$, and let $P:X\to X$ be a continuous polynomial given by $P((x_i)_i):=(p(x_{i+m}))_i$ where $p:C\to C$ is a polynomial of degree strictly greater than 1 such that $p(0)=0$. If 0 is repelling then P is quasi D-chaotic.

Proof: Since repelling fixed points are contained in the Julia set. In view of theorem (2.7) we just have to show the quasi density of periodic points. Since 0 is a repelling fixed point, then $|p'(0)|>1$. Let $x\in X$ and $\varepsilon>0$ and select $|p'(0)|>\lambda>1$. Let $k\in N$ with $\|(x_1,\ldots,x_k,0,0,\ldots)\|<(\varepsilon/2^{1/q+1})$ and $\delta<\frac{\varepsilon}{6k2^{1/q}}$ such that $\lambda\mathcal{U}_0\subset p(\mathcal{U}_0)$ for any disk $\mathcal{U}_0$ centered at 0 of radius smaller than δ.

Since $0\in J(p)$ and $\{x_1,\ldots,x_k\}\subset C$ (by Th. 2.6), we find n>k and $\{z_{1,1},\ldots,z_{1,k}\}\subset C$ such that $|z_{1,i}|<\delta$ and $|p^n z_{1,i}-x_i|<\delta$, $i=1,2,\ldots,k$. Without loss of generality n is chosen so that $\sum_{t=1}^{\infty}\lambda^{-tn}<1$. Proceeding by induction we select $\{z_{j,i}\in C, j>1, i=1,\ldots,k\}$ satisfying $|z_{j+1,i}|<\lambda^{-jn}\delta$ and $p^n z_{j+1,i}=z_{j,i}$, $j\in N$, $i=1,\ldots,k$ we then defines z as $z:=(z_{0,1},\ldots,z_{0,k},0,\ldots,0,z_{1,1}^{mn+1},\ldots,z_{1,k},0,\ldots,0,z_{2,1}^{(m+1)n+1},\ldots)$, (where $z_{1,1}^{nm+1}$ (resp. $z_{2,1}^{(m+1)n+1},\ldots$) means that $z_{1,1}$ (resp. $z_{2,1},\ldots$) is in the m+1 (resp. (m+1)n+1,...)-position), where $z_{0,1}:=p^n z_{1,i}$, $i=1,\ldots,k$, then $P^n z:=(p^n z_{1,1},\ldots,p^n z_{1,k},0,\ldots,0,p^n z_{2,1},\ldots,p^n z_{2,k},0,\ldots)=(z_{0,1},\ldots,z_{0,k},0,\ldots,0,z_{1,1},\ldots,z_{1,k},0,\ldots)$.

Therefore z is a periodic point for P of period n. Now, take $\bar{x}:=(x_1,\dots,x_k,0,\dots)$ then ${}_q\|x-z\|\le c[{}_q\|x-\bar{x}\|+{}_q\|z-\bar{x}\|]$ when $c=2^{1/q}$.

Now, ${}_q\|z-\bar{x}\|=[\sum_{i=1}^{k}|z_{0,i}-x_i|^q+\sum_{i=1}^{k}|z_{1,i}|^q+\sum_{j=2}^{\infty}\sum_{i=1}^{k}|z_{j,i}|^q]^{1/q}\le[\sum_{i=1}^{k}|p^n z_{1,i}-x_i|^q+k\delta^q+k\sum_{t=1}^{\infty}(\lambda^{-tn}\delta)^q]^{1/q}\le[k\delta^q+k\delta^q+k\sum_{t=1}^{\infty}(\lambda^{-tn}\delta)^q]^{1/q}\le k^{1/q}\delta+k^{1/q}\delta+k^{1/q}\sum_{t=1}^{\infty}\lambda^{-tn}\delta \le k^{1/q}\frac{\varepsilon}{6k2^{1/q}}+k^{1/q}\frac{\varepsilon}{6k2^{1/q}}+k^{1/q}\frac{\varepsilon}{6k2^{1/q}}\sum_{t=1}^{\infty}\lambda^{-tn}$

$=2k^{1/q}\frac{\varepsilon}{6k2^{1/q}}+k^{1/q}\frac{\varepsilon}{6k2^{1/q}}\sum_{t=1}^{\infty}\lambda^{-tn} < 3k^{1/q}\frac{\varepsilon}{6k2^{1/q}}+k^{1/q}\frac{\varepsilon}{6k2^{1/q}} < \frac{\varepsilon}{3(2^{1/q})}+\frac{\varepsilon}{6(2^{1/q})}$ (since $\frac{k^{1/q}}{k}\le 1, \sum_{t=1}^{\infty}\lambda^{-tn}<1$). Therefore

$${}_q\|x-z\|\le 2^{1/q}\left[\frac{\varepsilon}{2(2^{1/q})}+\frac{\varepsilon}{3(2^{1/q})}+\frac{\varepsilon}{6(2^{1/q})}\right]=\frac{\varepsilon}{2}+\frac{\varepsilon}{3}+\frac{\varepsilon}{6}=\varepsilon$$

Thus the set of periodic points is quasi dense in X and hence P is quasi D-Chaotic.

Remark(2.4): It is well known that on any quasi-Banach space X. If a continuous polynomial P^n for some $n\in N$ is a quasi hypercyclic then so is P, because if P^n is quasi hypercyclic on a quasi-Banach space X, there exist $x\in X$ such that $orb(P^n,x)=\{(P^n)^k x:k\ge 0\}$ is a quasi dense in X. But $\{(P^n)^k x:k\ge 0\}\subset\{P^k x:k\ge 0\}$, therefore $\{(P^n)^k x:k\ge 0\}$ is a quasi dense in X, and thus x is quasi hypercyclic vector for P, hence P is quasi hypercyclic polynomial.

The following theorem shows that the converse is also true on the space ℓ_q , $0<q<1$.

Theorem(2.9):Let $X=\ell_q$, $0<q<1$, and let $P:X\to X$ be a continuous polynomials given by $P((x_i)_i):=(p(x_{i+m}))_i$ where $p:C\to C$ is a polynomial of degree greater than 1 such that $p(0)=0$. If P is quasi hypercyclic, then P^n is also quasi hypercyclic for each $n\ge 2$.

Proof: Since P is quasi hypercyclic on ℓ_q , $0<q<1$, then (by Th. 2.7) $0\in J(p)$, and since $J(p)=J(p^n)$ $\forall n\ge 2$ (by Th. 2.5) thus (by Th. 2.7) P^n is quasi hypercyclic for each $n\ge 2$.

Theorem(2.10): Let $X=\ell_q$, $0<q<1$ and let $P:X\to X$ be a continuous polynomial given by $P((x_i)_i):=(p(x_{i+m}))_i$ where $p:C\to C$ is polynomial of degree strictly greater than 1 such that $p(0)=0$. If P is quasi hypercyclic then so is λP for each $\lambda\in R^+$ with $\lambda\ge 1$.

Proof: Since P quasi hypercyclic polynomial then from theorem (2.7) $0\in J(p)$. Thus (by Prop. 2.8) $0\in J(\lambda p)$, $\lambda\in R^+$ with $\lambda\ge 1$, then (by Th. 2.7) we get that λP is quasi hypercyclic.

Note that by the same proof of the theorem 2.10 (resp. of theorem 2.9) we can show that if P is quasi AY-chaotic on ℓ_q, $0<q<1$ thenλP (resp. P^n $\forall n\ge 2$) is also quasi AY-chaotic for each $\lambda\in R^+$ with $\lambda\ge 1$.

In the following proposition we consider the weighted polynomials on quasi-Banach space ℓ_q , $0<q<1$.

Proposition(2.9): Let $X=\ell_q$, $0<q<1$ and let $P:X\to X$ be a continuous polynomial given by $P((x_i)_i):=(p(x_{i+m}))_i$ where $p:C\to C$ is polynomial of degree strictly greater than 1 such that $p(0)=0$. If P is quasi hypercyclic then P has quasi sensitive dependence on initial conditions.

Proof: For $\varepsilon=1$, given $x\in\ell_q$ and $\delta>0$, then we can find $k\in N$ such that ${}_q\|x-(x_1,\dots,x_k,0,0,\dots)\|<\delta$. Since P is a quasi hypercyclic, then (by Remark 2.3) there exist $z\in\ell_q$ quasi hypercyclic vector for P such that ${}_q\|x-z\|<\delta$.

Now $\bar{x}=(x_1,\dots,x_k,0,0,\dots)$

$P\bar{x}=(\frac{1}{2}p(x_2),\frac{1}{3}p(x_3),\dots)$

$P^2\bar{x}=(\frac{1}{2}p(\frac{1}{3}p(x_3)),\frac{1}{3}p(\frac{1}{4}p(x_4)),\dots)$

$P^3\bar{x} = (\frac{1}{2}p(\frac{1}{3}p(\frac{1}{4}p(x_4))), \frac{1}{3}p(\frac{1}{4}p(\frac{1}{5}p(x_5))), \ldots)$

$\vdots$

$P^k\bar{x} = (\frac{1}{2}p(\frac{1}{3}p(\ldots\frac{1}{k+1}p(x_{k+1})\ldots),\ldots)$.

And since p(0)=0 then $P^k(\bar{x})=0$ since $(x_{k+1}=0)$ thus $P^n(\bar{x})=0$, $\forall n>k$ and then ${}_q\|P^n\bar{x}\|$, $\forall n>k$. Since z is quasi hypercyclic vector for P, then (by Prop. 2.4) $\sup\{{}_q\|P^n z\|: n\geq 0\}=\infty$, then we get $n\in N$ satisfying: ${}_q\|P^n z\| > 2^{\frac{2}{q}+1}$ Then $2^{\frac{2}{q}+1} < {}_q\|P^n z\| \leq c[{}_q\|P^n z - P^n\bar{x}\| + {}_q\|P^n\bar{x}\|] = 2^{\frac{1}{q}}[{}_q\|P^n z - P^n\bar{x}\| + 0]$, then $2^{\frac{1}{q}+1} < {}_q\|P^n z - P^n\bar{x}\| \leq 2^{\frac{1}{q}}[{}_q\|P^n x - P^n z\| + {}_q\|P^n\bar{x} - P^n x\|]$, thus $2 < {}_q\|P^n x - P^n z\| + {}_q\|P^n\bar{x} - P^n x\|$, and since $\varepsilon=1$ then ${}_q\|P^n\bar{x} - P^n x\| > \varepsilon$ or ${}_q\|P^n x - P^n z\| > \varepsilon$. Hence P has quasi sensitive dependence on initial conditions.

Also we can prove that in proposition (2.9) if the polynomial P given by $P((x_i)_i) := (w_i p(x_{i+1}))$ where $\{w_i : i\in Z\}$ be a bounded sequence of real numbers, then P has quasi sensitive dependence on initial conditions if P is quasi hypercyclic.

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