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Semigroup Theory for Dual Dynamic Programming

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Abstract:
In this paper, the nonclassical approach to dynamic programming for the optimal control problem via strongly continuous semigroup has been presented. The dual value function V_D (.,.) of the problem is defined and characterized. We find that it satisfied the dual dynamic programming principle and dual Hamilton –Jacobi – Bellman equation. Also, some properties of V_D (. , .) have been studied, such as, various kinds of continuities and boundedness, these properties used to give a sufficient condition for optimality. A suitable verification theorem to find a dual optimal feedback control has been proved. Finally gives an example which illustrates the value of the theorem which deals with the sufficient condition for optimality.

Keywords: Dual value function, Dual dynamic programming, semigroup theory, HJB equation, Dual optimal feedback control.

نظرية شبة الزمرة للبرمجة الديناميكية المواجهة

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الخلاصة:
في هذا البحث، تم تقديم اسلوب غير تقليدي للبرمجة الديناميكية لمسألة السيطرة المثلى بواسطة شبة الزمرة المستمرة بقوة. حيث عرفت دالة القيمة المواجهة $V_D(. , .)$ لهذه المسألة مع ذكر خصائصها، حيث وجدنا بان هذه الدالة تحقق مبدأ البرمجة الديناميكية المواجهة ومعادلة هاملتون جاكوبي –بلمان. كذلك تم دراسة بعض الخواص لدالة القيمة المواجهة أمثال أنواع متنوعة من الاستمرارية والحدودية ومن ثم استخدام هذه الخواص لإثبات المبرهنة التي تتعامل مع الشرط الكافي للامثلية. ايضا تم إثبات نظرية تحقيق مناسبة لإيجاد مسيطر تغذية استرجاعية مثلى مواجهة. وأخيرا تم إعطاء مثال يوضح قيمة النظرية التي تتعامل مع الشرط الكافي للامثلية.

Introduction:

The theory of semigroup of linear operators lends a convenient setting and offers many advantages for applications. Control theory in infinite dimensional spaces is a relatively new field and started blooming only after well –developed semigroup theory was at hand [1,2]. Many scientific, engineering and economics problems can be modeled by partial differential equations, integral equations can be described as differential equations or differential inclusion [2,3].

A basic topic in optimal control is the analysis and applications of a specific value function, namely, the minimum cost in an optimal control problem as a function of the starting time and state (fixed initial point). Dynamic programming (briefly DP)is a branch of optimal control theory that deals with such a value function, i.e., whenever this value function is a Lipschitz solution for the partial

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differential equation of DP, known as the Hamilton –Jacobi –Bellman (briefly, HJB) equation, then it satisfies the sufficient conditions for optimality [2,4-9].

The problem to be investigated in this paper is the following optimal control problem. We will consider the following state equation:

$$\dot{z}(t) = A\, z(t) + f\,(t, z(t), u(t)),\ \ t \in [0, T]$$
$$z(0) = x, \tag{1.1}$$

where $A : D(A) \subset X \to X$ is the generator of some strongly continuous semigroup of linear bounded operators (briefly, C_o semigroup) $\{e^{At}\}_{t\geq 0}$ on a separable Hilbert space X, with $X^{*}= X$, * denoted the dual, where $f :[0, T] \times X \times U \to X$ is a given map with U a metric space in which the control $u(t)$ take values.

Thus, for any initial state $x \in X$ and control $u(.) \in$ U$[0, T] \equiv \{ u : [0, T]\to U \mid u(.)$ measurable $\}$, the corresponding trajectory $z(.)$ is the mild solution of (1.1) given by [10]:

$$z(t) = e^{At}x + \int_0^t e^{A(t-s)} f(s, z(s), u(s))\, ds,\ t \in [0, T\,]. \tag{1.2}$$

We will assume that f is Lipschitz continuous in z, uniformly in $(t, u) \in [0,T] \times U$. Thus, the (mild) solution to (1.1) is uniquely determined by the initial state and the control [10].

Our cost functional is given by the following

$$J(u(.)) = \int_0^T f^{o}(t, z(t), u(t))\, dt + h(z(T), \tag{1.3}$$

where $f^{o}: [0,T]\times X \times U \to \mathbb{R}$ and $h: X \to \mathbb{R}$ are given functions. And the optimal control problem is stated as follows:

Problem (C). Find $\bar{u}(.) \in$ U$[0,T]$, such that

$$J(\bar{u}(.)) = \inf_{u(.)\,\in\, \mathcal{U}[0,T]} J(u(.)). \tag{1.4}$$

Now, let us describe the DP method. Instead of considering Problem (C) with (1.2) and (1.3), we consider the following family of optimal control problems:

For any given $(t,x) \in [0, T] \times X$, let us consider the following state equation

$$z_{t,x}(s) = e^{A(s-t)}x + \int_t^s e^{A(s-r)} f(r, z_{t,x}(r), u(r))\, dr,\ s \in [t,T\,], \tag{1.5}$$

with $u(.) \in$ U$[0,T]$ and the cost functional

$$J_{t,x}(u(.)) = \int_t^T f^{o}(r, z_{t,x}(r), u(r))\, dr + h(z_{t,x}(T)). \tag{1.6}$$

Here, the subscripts t and x are used to emphasize the dependence of the trajectory and the cost functional on the initial condition (t, x). Next, we define the function $V : [0,T] \times X \to \mathbb{R}$ by the following :

$$V(t, x) = \inf_{u(.)\,\in\, \mathcal{U}[t,T]} J_{t,x}(u(.)),\quad V(T, x) = h(x). \tag{1.7}$$

The function V is called the valued function of Problem (C).

In [2, Chapter 6] the author devoted to the study of another important approach to optimal control problem, he introduced the DP method for problem (1.1). For problem (1.1), but when $f(., ., .) \equiv Bu(.)$ where the control operator B is linear and unbounded, Faggian [11, 12], applied the DP to show that the value function of the problem is a solution of an integral version of the HJB equation. Also Faggian [13] applied DP to show that the value function of economic problem is the unique strong solution of the associated HJB equation. The classical and dual DP for finite dimensional optimal control problem of Bolza have been introduced in [6, 14], if we used the dual DP [14] we need not require that the value function is differentiable, which is essential in the classical method [6]. In [4] the authors applied the dual DP to prove that, the existence of a maximum solution to HJ equation for the Lagrange problem. Also, the existence of a minimum solution to the dual partial differential equation of DP for optimal control problems of Bolza and Lagrange have been proved by using dual DP in[5].

From all the above one can find a reasonable justification to accomplish the study of this paper. Thus, the aim of this paper is to describe the nonclassical approach to DP via semigroup theory for optimal control Problem (C). The dual value function of this problem is defined, and we show that this function satisfied the dual DP principle and dual HJB equation. Another properties of dual value function have been studied, these properties used to give a sufficient condition for optimality. Also a verification theorem to find an optimal state feedback control via dual value function have been proved.

The method used in this paper is completely in the spirit of DP technique, although it is quite new in the study of the value function.

Definitions and Theorems:

Before proceeding to main results, we shall set in this section some definitions and theorems that will be used in our subsequent discussion.

Definition 2.1 **[2]:** Let X be a normed space. Then the set of all bounded linear functionals (or linear continuous functional) on X constitutes a normed space with norm defined by

$$\| f \| = \sup(| f(x) | / \|x\| : x \in X, x \neq 0) = \sup(| f(x) | : x \in X, \|x\| = 1)$$

Theorem 2.1**[15]:** Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0). Then the change

$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$, in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

In which each of $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.

Corollary 2.1 **[9]:** If $0 \leq m(t) \leq D + C \int_s^t m(r)\, dr$ for $s \leq t \leq T$ where C, D are nonnegative constants. Then:

$$m(t) \leq D e^{c(t-s)}, \quad s \leq t \leq T. \tag{2.1}$$

Inequality (2.1) is called **Gronwell's inequality**.

Definition 2.2 **[10]:** Let X be a Banach space. A one parameter family $T(t), 0 \leq t < \infty$, of bounded linear operators from X into X is ***a semigroup*** of bounded linear operators on X if

i. $T(0) = I$, (I is the identity operator on X),

ii. $T(t+s) = T(t) + T(s)$ for every $t, s \geq 0$ (the semigroup property).

A semigroup of bounded linear operators, $T(t)$, $t \geq 0$ is ***uniformly continuous*** if

$$\lim_{t \to 0^+} \| T(t) - I \| = 0, \tag{2.2}$$

Definition 2.3 **[10]:** A semigroup $T(t)$, $0 \leq t < \infty$, of bounded linear operators on X is a ***strongly continuous semigroup*** of bounded linear operators if

$$\lim_{t \to 0^+} T(t)x = x \quad \text{for every} \quad x \in X. \tag{2.3}$$

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_o or simply a **C_o semigroup.**

Definition 2.4 **[1]:** The ***infinitesimal generator*** A of a C_o- semigroup on a Hilbert space X is defined by

$$A z = \lim_{t \to 0^+} \frac{1}{t} (T(t) - I) z \tag{2.4}$$

Whenever the limit exists, the domain of A, $D(A)$, being the set of elements in X for which the limit exists.

***Corollary 2.2* [10]:** Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators. Then (*a*) There exists a constant $\omega \geq 0$ s.t $\| T(t) \| \leq e^{\omega t}$. (*b*) There exists a unique bounded linear operator A s.t $T(t) = e^{At}$. (*c*) The operator A in (*b*) is the infinitesimal generator of $T(t)$. (*d*) $t \to T(t)$ is differentiable in norm and $d\,T(t)/dt = AT(t) = T(t)\,A$.

***Example 2.1* [1]:** Let X be a separable Hilbert space, the mapping given by

$$T(t)x = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle x, \varphi_n \rangle \varphi_n,$$

is a C_o semigroup on X, where $\{ \varphi_n, n \geq 1\}$ be an orthonormal basis in X, and $\{\lambda_n, n \geq 1\}$ be a sequence of complex numbers with $sup\, Re\{\lambda_n\} < \infty$.

The infinitesimal generator is $A\,x = \sum_{n=1}^{\infty} \lambda_n \varphi_n \langle \varphi_n, x \rangle$, with the domain

$$D(A) = \{ x : \sum_{n=1}^{\infty} \left| \lambda_n \langle \varphi_n, x \rangle \right|^2 < \infty \}.$$

***Example 2.2*[2]:** Let $A \in \mathbb{R}^{n\times n}$ be an $(n\times n)$- matrix. Then $T(t) = e^{At} = \sum_{k=0}^{\infty} (A^k t^k / k!),\ t \geq 0,$ is a C_o semigroup on $X = \mathbb{R}^n$. It is well known that e^{At} is the fundamental matrix of the (homogeneous) ordinary differential equation: $\dot{z}(t) = A\,z(t)$.

***Definition 2.5* [2]:** Let X be a Banach space. Suppose $\varphi : X \to \mathbb{R}$ and $x_0 \in X$. We define

$$D^{+}\varphi(x_0) = \{ y \in X^{*} |\ \overline{\lim_{x\to x_0}}\ \frac{\varphi(x) - \varphi(x_0) - \langle y, x - x_0 \rangle}{x - x_0} \leq 0 \}.$$

We call $D^{+}\varphi(x_0)$ the superdifferential of φ at x_0.

Properties of the value function:

Let us take the problem (1.1) in section 1, the goal of this section is to characterize the value function $V(.,.)$ in (1.7). The first result is the following theorem, which is called the DP principle.

***Theorem 3.1* [2]:** Let $(t, x) \in [0,T]\times X$. Then, for any $s \in [t,T]$,

$$V(t,x) = \inf_{u(.)\, \in\, \mathcal{U}[t,s]} \{ \int_t^s f^{o}(r, z_{t,x}(r), u(r))\, dr + V(s, z_{t,x}(s)) \}.$$

The next goal is to derive the so-called HJB equation for the value function V.

***Proposition 3.1* [2]:** Let the value function V be $C^1([0,T]\times X)$. Let the functions f, f^{o} and h be continuous. Then V satisfies the following HJB equation:

$$V_t + \langle V_x, A\,x \rangle + H(t, x, V_x) = 0,\ \ (t,x) \in [0,T]\times D(A),$$

$$V\big|_{t=T} = h(x), \qquad x \in X,$$

Where $H(t,x,q) = \inf_{u\in U} \{ \langle q, f(t,x,u) \rangle + f^{o}(t,x,u) \}$, $(t, x, q) \in [0, T] \times X \times X$.

For more details about properties of the value function one can refer to [2,7,11,12,13].

Semigroups for dual dynamic programming:

In this section we suggest the nonclassical approach to DP for the control problem (1.1) via semigroup theory, the domain of exploration was carried out from the (t, x)-space to the space of multipliers ((t,y^0,y)-space). We will define the dual value function for the problem (1.1), and study some properties of this function such as, the dual value function is a solution to the dual Hamilton –

Jacobi –Bellman (DHJB) equation, which is essential in study the optimality. Finally, a suitable verification theorem is proved.

The Dual Value Function:

Let us take the optimal control problem as introduced in section 1. And we define an admissible pair for these problem as follows.

***Definition 4.1*: For the problem (1.1), any pair $(z(.),u(.)) \in C([0,T]; X) \times U[0,T]$ satisfying (1.2) is called an admissible pair.

Now, let $K \subset \mathbb{R} \times X$ denoted a set covered by the graphs of all admissible trajectories for the problem (1.1). And let $P \subset \mathbb{R}^2 \times X$ be a set of variables $(t, y^0, y) = (t, p)$, $t \in [0,T]$, with $y^0 \le 0$ and nonempty interior. Take a function $z(t,p)$ defined on P such that $(t, z(t,p)) \in K$, $(t,p) \in P$, we assume that it is measurable, locally bounded and that for each admissible trajectory $z(t)$ lying in K, there exists an absolutely continuous function $p(t) = (y^0, y(t))$ lying in P such that $z(t) = z(t, p(t))$, and if all trajectories $z(t)$ start at the same (t_0, x_0), then all the corresponding $p(t)$ have the same first coordinate y^0.

Thus, we can define the real value function $V_D(t,p)$ in a set $P \subset \mathbb{R} \times X$ of the dual space $(t, y^0, y) = (t, p)$, $y^0 \le 0$ as follows:

$$V_D(t,p) = \inf\{-y^0 \int_t^T f^o(r, z_{t,x}(r), u(r))\,dr - y^0\, h(z_{t,x}(T))\}$$

$$= \inf\{-y^0 J_{t,x}(u(.))\}, \qquad (4.1)$$

where the infimum above taken over admissible pairs $z(\tau)$, $u(\tau)$, $\tau \in [t,T]$, whose trajectories start at $(t, z(t,p))$, and

$$V_D(T,p) = -y^0\, h(z_{t,x}(0), p(0)) = -y^0\, h(x).$$

The function V_D is called the dual value function of the Problem (C) which is discussed in section 1.

***Remark 4.1* :** Let $V_D(t,p)$ be as in (4.1) but with K and $z(t,p)$ defined above. Then we see that

$$V_D(t,p) = -y^0\, V(t, z(t,p)), \quad (t,p) \in P,\ y^0 \le 0.$$

Following, we get a modification of Th. 3.1, and Prop. 3.1. Thus, the goal of this section is to characterize the dual value function. Our first result is the following theorem, which is called the dual DP principle.

***Theorem 4.1* :** Let $z(t) = z(t, p(t)), (t,p) \in P,$ where $(t, z(t,p)) \in K = [0,T] \times X$. Then for any $s \in [t,T]$,

$$V_D(t,p) = \inf\{-y^0 \int_t^T f^o(r, z_{t,x}(r), u(r))\,dr + V_D(s, z_{t,x}(s))\}, \qquad (4.2)$$

Where the infimum above is taken over admissible pairs $z(\tau)$, $u(\tau)$, $\tau \in [t,T]$, whose trajectories start at $(t, z(t,p))$.

Before proving the above theorem, let us first make some observations on (4.2). Suppose (4.2) holds and for a given $(t, z(t,p)) \in K$, there exists an optimal control $\bar{u}(.)$, and $\bar{z}_{t,x}(.) = \bar{z}_{t,z(0,p(0))}(., \bar{p}(.))$ is the corresponding optimal trajectory. Then

$$V_D(t,p) = -y^0 \int_t^T f^o(r, \bar{z}_{t,x}(r,p), \bar{u}(r))\,dr - y^0\, h(\bar{z}_{t,x}(T,p))$$

$$=-y^0 \int_t^s f^o(r,\bar{z}_{t,x}(r,p),u(r))\,dr - y^0 \int_s^T f^o(r,\bar{z}_{s,x}(r,p),u(r))\,dr - y^0 h(\bar{z}_{s,x}(T,p))$$

$$=-y^0 \int_t^s f^o(r,\bar{z}_{t,x}(r,p),u(r))\,dr - y^0 J_{s,\bar{z}_{t,x}(s)}(\bar{u}|_{[s,T]}(.))$$

$$\geq -y^0 \int_t^s f^o(r,\bar{z}_{t,x}(r,p),\bar{u}(r))\,dr + V_D(s,\bar{z}_{t,x}(s,p)) \quad (4.3)$$

$$\geq \inf\{-y^0 \int_t^s f^o(r,\bar{z}_{t,x}(r,p),u(r))\,dr + V_D(s,z_{t,x}(s,p))\} = V_D(t,p)$$

Where the infimum above is taken over admissible pairs $z(\tau)$, $u(\tau)$, $\tau \in [t,T]$, whose trajectories start at $(t, z(t,p))$.

Therefore, the equalities in the middle of (4.3) hold. This implies that

$$-y^0 J_{s,\bar{z}_{t,x}(s)}(\bar{u}|_{[s,T]}(.)) = V_D(s,\bar{z}_{t,x}(s,p)).$$

In other word, $\bar{u}|_{[s,T]}(.)$ is an optimal control of the problem starting from $(s,\bar{z}_{t,x}(s))$ = $(s,\bar{z}_{t,x}(s,\bar{p}(s)))$ with the optimal trajectory $\bar{z}_{t,x}|_{[s,T]}(.) = \bar{z}_{t,x}(.,\bar{p}(.))|_{[s,T]}(.)$. This says that

Globally optimal → locally optimal,

Which is the essence of the DP method.

Now, let us given a proof of Theorem 4.1

Proof of Theorem 4.1. First of all, for any $u(.) \in \mathcal{U}[s,T]$ and any $u(.) \in \mathcal{U}[t,s]$, by putting Theorem concatenatively, we obtain $u(.) \in \mathcal{U}[t,T]$. Thus by definition of the dual value function we get that

$$V_D(t,p) \leq -y^0 \int_t^T f^o(r,z_{t,x}(r,p),u(r))\,dr - y^0 h(z_{t,x}(T,p))$$

$$=-y^0 \int_t^s f^o(r,z_{t,x}(r,p),u(r))\,dr - y^0 \int_s^T f^o(r,z_{t,x}(r,p),u(r))\,dr - y^0 h(z_{s,x}(T,p))$$

$$=-y^0 \int_t^s f^o(r,z_{t,x}(r,p),u(r))\,dr - y^0 J_{s,z_{t,x}(s,p)}(u(.)),$$

by taking the infimum over admissible pairs $z(\tau)$, $u(\tau)$, $\tau \in [s, T]$, whose trajectories start at $(s, z(s,p)) \in K$, we obtain

$$V_D(t,p) \leq -y^0 \int_t^s f^o(r,z_{t,x}(r,p),u(r))\,dr + V_D(s,z_{t,x}(s,p)).$$

Consequently,

$$V_D(t,p) \leq \Omega(t,s,x) \equiv \text{The right –hand side of (4.2).}$$

Next, for any $\varepsilon > 0$, there exists a $u^{\varepsilon}(.) \in \mathcal{U}[t,T]$, such that

$$V_D(t,p)+\varepsilon \geq -y^0 J_{t,x}(u^{\varepsilon}(.))$$

$$= -y^0 \int_t^s f^o(r, z_{t,x}(r,p), u^{\varepsilon}(r))\,dr - y^0 J_{s, z_{t,x}(s)}(u^{\varepsilon}(.))$$

$$\geq -y^0 \int_t^s f^o(r, z_{t,x}(r,p), u^{\varepsilon}(r))\,dr + V_D(s, z_{t,x}(s,p)) \geq \Omega(t,s,x).$$

Hence, (4.2) follows. □

Our next goal is to derive the dual HJB equation for the dual value function V_D.

Proposition 4.2: Suppose that $z(t) = z(t, p(t)), (t,p) \in P \subset \mathbb{R}^2 \times X$ where $(t, z(t,p)) \in K = [0,T] \times X$. Let the dual value function $V_D \in C^1([0,T]\times(-\infty,0]\times X)$, and let the functions *f*, f^o and *h* be continuous. Then V_D satisfies the following dual HJB equation

$$V_{Dt} + \langle V_{Dx}, Ax\rangle + H(t,x,V_{Dx}) = 0, \quad (t,x) \in [0,T]\times D(A),$$
$$V_D\big|_{t=T} = -y^0 h(x), \quad x \in X \tag{4.4}$$

Where

$$H(t,x,v) = \inf_{u\in U} \{\langle v, f(t,x,u)\rangle - y^0 f^o(t,x,u)\}, \quad (t,x,v) \in [0,T]\times X\times X$$

Proof: First, by definition, $V_D(T,p) = -y^0 h(x)$ is satisfied.

Next, let us fix a $u \in U$ and $x \in D(A)$ (the domain of the generator of C_o semigroup e^{At} on a separable Hilbert space X). By (4.2) and Th. 2.1, we have that

$$0 \leq V_D(s, z_{t,x}(s)) - V_D(t,p) - y^0 \int_t^s f^o(r, z_{t,x}(r,p), u(r))\,dr$$

$$= V_D(s, z_{t,x}(s,p)) - V_D(t,p) - y^0 \int_t^s f^o(r, z_{t,x}(r,p), u(r))\,dr$$

$$= V_{Dt}(t,p)(s-t) + \langle V_{Dx}(t,p), z_{t,x}(s,p) - x\rangle$$

$$- y^0 \int_t^s f^o(r, z_{t,x}(r,p), u(r))\,dr + o(|s-t| + |z_{t,x}(s,p)-x|). \tag{4.5}$$

We not that because $x \in D(A)$, we see that

$$\frac{1}{s-t}(z_{t,x}(s,p)-x) = \frac{1}{s-t}(e^{A(s-t)} - I)x + \frac{1}{s-t}\int_t^s e^{A(s-r)} f(r, z_{t,x}(r,p), u)\,dr$$
$$\to Ax + f(t,x,u) \quad \text{as} \quad s \downarrow t. \tag{4.6}$$

Hence, dividing by $(s-t)$ in (4.5) and sending $s \downarrow t$, we obtain that

$$0 \leq V_{Dt}(t,p) + \langle V_{Dx}(t,p), Ax + f(t,x,u)\rangle - y^0 f^o(t,x,u), \ \forall u \in U.$$

Thus, it follows that

$$0 \leq V_{Dt}(t,p) + \langle V_{Dx}(t,p), Ax\rangle + H(t,x,V_{Dx}(t,p)). \tag{4.7}$$

On the other hand, let $x \in D(A)$ be fixed. For any $\varepsilon > 0$ and $s > t$, by (4.2), there exists a $\tilde{u}(.) \equiv u^{\varepsilon, s}(.) \in \mathcal{U}[t,s]$, such that

$$\varepsilon(s-t) \geq V_D(s, z_{t,x}(s,p)) - V_D(t,p) - y^0 \int_t^s f^o(r, z_{t,x}(r,p), \tilde{u}(r))\, dr$$

$$= V_{Dt}(t,p)(s-t) + \langle V_{Dx}(t,p), (e^{A(s-t)} - I)\, x \rangle$$

$$+ \langle V_{Dx}(t,p), \int_t^s f(r, z_{t,x}(r,p), \tilde{u}(r))\, dr \rangle - y^0 \int_t^s f^o(r, z_{t,x}(r,p), \tilde{u}(r)) dr + o(|s-t|)$$

$$= V_{Dt}(t,p)(s-t) + \langle V_{Dx}(t,p), (e^{A(s-t)} - I)\, x \rangle + \int_t^s \{\langle V_{Dx}(t,p), f(t,x,\tilde{u}(r))\rangle$$

$$- y^0 f^o(t,x,\tilde{u}(r))\}\, dr + o(|s-t|)$$

$$\geq V_{Dt}(t,p)(s-t) + \langle V_{Dx}(t,p), (e^{A(s-t)} - I)\, x \rangle + H(t,x,V_{Dx}(t,p))(s-t) + o(|s-t|).$$

Then, dividing through by $(s-t)$ and letting $s, t \to 0$, we get that

$$\varepsilon \geq V_{Dt}(t,p) + \langle V_{Dx}(t,p), Ax \rangle + H(t,x,V_{Dx}(t,p)).$$

Combining with (4.7). We obtain the desired result. □

Remark 4.2: We derive the dual HJB equation (4.4) by assuming the dual value function V_D to be $C^1([0,T] \times X)$. This assumption, however, is not necessarily true in most cases. We will provide an example below to illustrate this point. Hence, the conclusion of Prop. 4.2 has lack of applicability. Thus we can introduce proper notions of solutions to the equation (4.4) so that the dual value function V_D is the unique "solution" of (4.4).

To conclude this section, let us present an example where the dual value function is not in $C^1([0,T] \times X)$.

***Example 4.1*:** Consider in ℝ the following system:

$$\dot{z}_{t,x}(s,p) = u(s)\, z_{t,x}(s,p), \qquad s \in [t,T],$$

$$z_{t,x}(t,p) = z(0, p(0)) = x,$$

with the control domain $U = [0,1]$ and the cost functional

$$J_{t,x}(u(.)) = z_{t,x}(T,p).$$

Then it is not hard to see that the dual value function is given by

$$V_D(t, z(0,p)) = V_D(t,x) = \begin{cases} -y^0 z(0,p(0)) = -y^0 x, x \geq 0 \text{ and } y^0 < 0 \\ \\ -y^0 z(0,p(0))\, e^{T-t} = -y^0 x\, e^{T-t}, \ x < 0, y^0 < 0. \end{cases}$$

Cleary, $V_D(t,x)$ is just Lipschitz continuous and is not C^1.

Remark 4.3: If we define the dual value function V_D in (4.1) as follows

$$V_D(t,p) = \sup\{ y^0 \int_t^T f^o(r, z_{t,x}(r), u(r))\, dr + y^0\, h(z_{t,x}(T))\}, \tag{4.8}$$

Then by using simple change in the proof of Th.4.1, we can obtain the following result.

$$V_D(t,p) = \sup\{ y^0 \int_t^T f^o(r, z_{t,x}(r), u(r))\, dr + V_D(s, z_{t,x}(s))\}.$$

Also, for Prop. 4.2, we see that the dual value function (4.8) satisfies the (DHJB) equation (4.4) but, where

$$H(t,x,v) = \sup_{u\in U} \{ y^0\, f^o(t,x,u) - \langle v, f(t,x,u)\rangle \}, (t, x, v) \in [0,T]\times X\times X, \text{ and}$$

$$V_D\big|_{t=T} = y^0\, h(x), \quad x \in X.$$

***Properties of the dual value function**:*

We present some basic properties of the dual value function associated with our optimal control problem. As in previous sections and hereafter, we let X be a separable Hilbert space with the inner product $\langle .,.\rangle$ and the induced norm $|\ .\ |$. We also let U be a metric space in which the control takes values.

We first study the continuity of dual value functions. In what follows, by a ***modulus of continuity***, we mean a continuous function $W:\mathbb{R}^+\to\mathbb{R}^+$, with $W(0) = 0$ and subadditive : $W(\sigma_1+\sigma_2) \le W(\sigma_1) + W(\sigma_2)$, for all $\sigma_1, \sigma_2 \ge 0$; by a ***local modulus of continuity***, we mean a continuous function. $W:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$, with the property that for each $r \ge 0$, $\sigma \mapsto W(\sigma, r)$ is a modulus of continuity. In what follows, in different places, W will represent a different (local) modulus of continuity.

Next, let us make the following assumptions:

(A_1) The linear, densely defined operator $A: D(A)\subset X\to X$ generates a C_o contraction semigroup e^{At} on the space X. Thus

$$\| e^{At} \| \le 1 \qquad \forall\ t \ge 0 \tag{4.9}$$

(A_2) $f:(-\infty,0]\times[0,T]\times X\times U\to X$ is continuous, such that for some constant $L > 0$ and local modulus of continuity W,

$$|f(t,z(t,p),u) - f(\bar t,\bar z(\bar t,\bar p),u)| \le L|z(t,p)-\bar z(\bar t,\bar p)| + W(|t-\bar t|, |z(t,p)|\vee|\bar z(\bar t,\bar p)|)$$

$$\forall\ (t,z(t,p)),(\bar t,\bar z(\bar t,\bar p)) \in K \,\&\, (t,p),(\bar t,\bar p) \in P,\ u\in U, \text{ (where } \vee \text{ denoted or)} \tag{4.10}$$

$$|f(t,0,u)| \le L, \quad \forall\ (t,u) \in [0,T]\times U \tag{4.11}$$

(A_3) $f^o:[0,T]\times(-\infty,0]\times X\times U\to\mathbb{R}$ And $h: X\to\mathbb{R}$ are continuous, and there exists a local modulus of continuity W such that

$$\left.\begin{aligned} &|f^o(t,z(t,p),u) - f^o(\bar t,\bar z(\bar t,\bar p),u)| \le \\ &W(|z(t,p)-\bar z(\bar t,\bar p)| + |t-\bar t|, |z(t,p)|\vee|\bar z(\bar t,\bar p)|) \\ &|h(z(t,p)) - h(\bar z(\bar t,\bar p))| \le \\ &W(|z(t,p)-\bar z(\bar t,\bar p)|, |z(t,p)|\vee|\bar z(\bar t,\bar p)|) \\ &\quad \forall\ (t,z(t,p)),(\bar t,\bar z(\bar t,\bar p)) \in K \,\&\, (t,p),(\bar t,\bar p)\in P, u\in U \end{aligned}\right\} \tag{4.12}$$

$$|f^o(t,0,u)|, |h(0)| \le L \quad \forall (t,u) \in [0,T]\times U, \tag{4.13}$$

For some constant $L > 0$ (here, we take it to the some as that in **(A_2)** just for simplicity).

(A_2)' In **(A_2)**, replace (4.10) by the following:

$$\begin{aligned} &|f(t,z(t,p),u) - f(\bar t,\bar z(\bar t,\bar p),u)| \le L(|z(t,p)-\bar z(\bar t,\bar p)| + |t-\bar t|) \\ &\forall (t,z(t,p)),(\bar t,\bar z(\bar t,\bar p)) \in K \,\&\, (t,p),(\bar t,\bar p)\in P, u\in U \end{aligned} \tag{4.14}$$

(A_3)' In **(A_3)** replace (4.12) by the following

$$\left.\begin{array}{l} \left| f^{o}(t,z(t,p),u)-f^{o}(\bar{t},\bar{z}(\bar{t},\bar{p}),u)\right| \le L(\left|z(t,p)-\bar{z}(\bar{t},\bar{p})\right|+\left|t-\bar{t}\right|) \\ \left|h(z(t,p))-h(\bar{z}(\bar{t},\bar{p}))\right| \le L\left|z(t,p)-\bar{z}(\bar{t},\bar{p})\right| \\ \forall (t,z(t,p)),(\bar{t},\bar{z}(\bar{t},\bar{p}))\in K \,\&\, (t,p),\ (\bar{t},\bar{p})\in P,\ u\in U \end{array}\right\} \quad (4.15)$$

Remark 4.4**[1]:** We known that for a general C_o semigroup e^{At} one always has $\| e^{At} \| \le M e^{\omega_0 t}$ for same $M \ge 1$and $\omega_0 \in \mathbb{R}$. As we are considering semilinear evolution equations, ω_0 can be taken to be 0, with out loss of generality. Thus, (4.9) is restrictive only in that $M = 1$. However, it is not hard to see that all the results in this subsection remain true for general cases.

It is clear that under (A_1) and (A_2), for any $(t,z(t,p))\in K \,\&\, (t,p)\in P$ and $u(.) \in \mathcal{U}[t,T]$, the state equation (1.5) admits a unique trajectory $z_{t,x}(.,p(.))$.

To study the boundedness and the continuity of the dual value function V_D, we first need to look at some properties of the trajectory $z_{t,x}(.,p(.))$. We collect these properties in the following lemma. In what follows, C is an absolute constant that can be different in different places.

Lemma 4.1**:** Suppose that $z(t) = z(t,p(t))$, $(t,p)\in P \subset \mathbb{R}^2 \times X$ where $(t,z(t,p)) \in K = [0,T] \times X$. Let ($A_1$) and ($A_2$) hold then for any $0\le t\le \bar{t} \le T$, $z(0,p(0))=x$, $\bar{z}(0,\bar{p}(0))=\bar{x} \in X$, and $u(.)\in \mathcal{U}[t,T]$, we have

$$\left|z_{t,x}(s,p)\right| \le C(1+|x|), \quad s\in[t,T] \quad (4.16)$$

$$\left|z_{t,x}(s,p)-z_{t,\bar{x}}(s,p)\right| \le C|x-\bar{x}|, \quad s\in[t,T] \quad (4.17)$$

$$\left|z_{t,x}(s,p)-z_{\bar{t},x}(s,p)\right| \le C\left|(e^{A(\bar{t}-t)}-I)x\right|+C(1+|x|)(\bar{t}-t),\ s\in[\bar{t},T] \quad (4.18)$$

$$\left|z_{t,x}(s,p)-e^{A(s-t)}x\right| \le C(1+|x|)(s-t), \quad s\in[t,T] \quad (4.19)$$

Proof: If we take $(t,z(t,p)) \in K$, then from (1.5) and (4.14) we have

$$\left|z_{t,x}(s,p)\right| \le |e^{A(s-t)}x| + \int_t^s \left| e^{A(s-r)} f(r,z_{t,x}(r,p),u(r))\right| dr$$

$$\le |x| + \int_t^s \left| f(r,z_{t,x}(r,p),u(r)) - f(r,0,u(r)) + f(r,0,u(r))\right| dr$$

$$\le |x| + \int_t^s \{L+\bar{L}|z_{t,x}(r,p)|\}dr \le L(s-t)+|x|+\bar{L}\int_t^s |z_{t,x}(r,p)|dr.$$

Thus, by Gronwell's inequality (2.1), we get that

$$\left|z_{t,x}(s,p)\right| \le (L(s-t)+|x|)e^{\bar{L}(s-t)} \le C(1+|x|),$$ Proving (4.16).

Now, from (1.5) and (4.14) we have

$$\left|z_{t,x}(s,p)-z_{t,\bar{x}}(s,p)\right| \le \left|e^{A(s-t)}x-e^{A(s-t)}\bar{x}\right|$$

$$+ \int_t^s \left| e^{A(s-r)}\right| \left| f(r,z_{t,x}(r,p),u(r)) - f(r,z_{t,\bar{x}}(r,p),u(r))\right| dr$$

$$\le |x-\bar{x}| + \int_t^s L\left| z_{t,x}(r,p) - z_{t,\bar{x}}(r,p)\right| dr$$

Thus, by Gronwell's inequality (2.1), we obtain that

$$\leq |x-\bar{x}|e^{L(s-t)} \leq C|x-\bar{x}|, \text{ proving } (4.17).$$

Now, we take $0 \leq t \leq \bar{t} \leq T$ and $x \in X$. From (1.5) and (4.16) we have.

$$\left|z_{t,x}(s,p)-z_{\bar{t},x}(s,p)\right| \leq \left|e^{A(s-t)}x-e^{A(s-\bar{t})}\right| + \int_t^{\bar{t}} L(1+\left|z_{t,x}(r,p)\right| + \int_{\bar{t}}^{s}\left|z_{t,x}(r,p)-z_{\bar{t},x}(r,p)\right|dr$$

$$\leq \left|(e^{A(\bar{t}-t)}-I)x\right| + L[1+C(1+|x|)](\bar{t}-t) + \int_{\bar{t}}^{s}\left|z_{t,x}(r,p)-z_{\bar{t},x}(r.p)\right|dr.$$

Thus, by Gronwell's inequality (2.1), we obtain (4.10). Finally, from (1.5) and (4.16) we have

$$\left|z_{t,x}(s,p) - e^{A(s-t)}x\right| \leq \int_t^s L(1+|z_{t,x}(r,p)|)dr \leq C(1+|x|)(s-t),$$

proving (4.19). □

We have seen that the estimates in (4.16)–(4.19) are uniform in the control $u(.)$. This is crucial in obtaining the properties of the dual value function $V_D(t,p)$. The next result continuous the local boundedness and various kinds of continuities of the dual value function.

Theorem 4.2: Suppose that $z(t) = z(t,p)$, $(t,p) \in P \subset \mathbb{R}^2 \times X$, where $(t, z(t,p)) \in K \subset [0,T] \times X$ and $z(0,p(0)) = x$, $\bar{z}(0,\bar{p}(0)) = \bar{x} \in X$. Now let ($A_1$)-($A_3$) hold. Then, for some increasing function $\hat{C}$ and some local modulus of continuity $\hat{W}$,

$$|V_D(t,p)| \leq |-y^0| \hat{C}|x|, \forall t \in [0,T], \quad x \in X, \ y^0 < 0. \tag{4.20}$$

$$|V_D(t,p)-V_D(t,\bar{p})| \leq |-y^0| \hat{W}(|\bar{x}-x|,|x| \vee |\bar{x}|) \ \forall t \in [0,T], x, \bar{x} \in X, y^0 < 0. \tag{4.21}$$

$$|V_D(t,p)-V_D(\bar{t},p)| \leq |-y^0| \hat{W}(|\bar{t}-t|+|(e^{A|\bar{t}-t|}-I)x|,|x|), \tag{4.22}$$

$$\forall \ t,\bar{t} \in [0,T], \ x \in X, \ y^0 < 0.$$

$$\left|V_D(t,p)-V_D(\bar{t},e^{A(\bar{t}-t)}x)\right| \leq |-y^0| \hat{W}(\bar{t}-t,|x|) \ \forall \, 0 \leq t \leq \bar{t} \leq T, x \in X, y^0 < 0. \tag{4.23}$$

Consequently,

$$\left|V_D(t,p)-(-y^0 h(e^{A(T-t)}x))\right| \leq |-y^0| \hat{W}(T-t,|x|) \ \forall t \in [0,T], x \in X, y^0 < 0. \tag{4.24}$$

In the case where (A_1), (A_2)' and (A_3)' hold, we have some constant $C > 0$ such that

$$|V_D(t,p)| \leq |-y^0| C(1+|x|), \ \forall \ (t,x) \in K, \ y^0 < 0. \tag{4.25}$$

$$|V_D(t,p)-V_D(t,\bar{p})| \leq |-y^0| C|x-\bar{x}| \ \ \forall x, \ \bar{x} \in X, y^0 < 0. \tag{4.26}$$

$$|V_D(t,p)-V_D(\bar{t},p)| \leq |-y^0| \ C((1+|x|)|\bar{t}-t| + |(e^{A|t-\bar{t}|}-I)x|), \tag{4.27}$$

$$\forall \ t, \ \bar{t} \in [0,T], x \in X.$$

$$\left|V_D(t,p)-V_D(\bar{t},e^{A(\bar{t}-t)}x)\right| \leq |-y^0| C(1+|x|)|t-\bar{t}|, \forall \, 0 \leq t \leq \bar{t} \leq T, x \in X, y^0 < 0. \tag{4.28}$$

Proof: For any $t \in [0,T], x, \bar{x} \in X, y^0 < 0,$ and any control $u(.) \in \mathcal{U}[t,T]$, by (4.12), (4.13) and (4.16) we have

$$V_D(t,p) \leq \left| -y^0 J_{t,x}(u(.)) \right|$$

$$= \left| \; -y^0 \int_t^T f^o(r, z_{t,x}(r,p), u(r))\, dr - y^0\, h(z_{t,x}(T,p) \; \right|$$

$$\leq \left| \; -y^0 \int_t^T \{ f^o(r, z_{t,x}(r,p), u(r)) - f^o(r,0,u(r)) + f^o(r,0,u(r)) \}\, dr \; \right|$$

$$+ \left| -y^0 [\, h(z_{t,x}(T,p) - h(0) + h(0) \,] \right|$$

$$\leq \left| -y^0 \right| \int_t^T \{ L + W(\left| z_{t,x}(r,p) \right|, \left| z_{t,x}(r,p) \right|) \}\, dr + \left| -y^0 \right| \{ L + W(\left| z_{t,x}(T,p) \right|, \left| z_{t,x}(T,p) \right|) \}$$

$$\leq \left| -y^0 \right| \int_t^T \{ L + W(C(1+|x|), C(1+|x|)) \}\, dr + \left| -y^0 \right| \{ L + W(C(1+|x|), C(1+|x|)) \}$$

$$= \left| -y^0 \right| \{ L + W(C(1+|x|), C(1+|x|)) \} \, (T-t) + \left| -y^0 \right| \{ L + W(C(1+|x|), C(1+|x|)) \}$$

$\leq (T+t) \left| -y^0 \right| \{ L + W(C(1+|x|), C(1+|x|)) \}$. This gives (4.20).

Now let $t \in [0,T], \;\; x, \;\; \bar{x} \in X, \;\; y^0 < 0$, and $u(.) \in \mathcal{U}[t,T]$, by (4.12), (4.16), (4.17) we have

$$\left| -y^0\, J_{t,x}(u(.)) - (-y^0 J_{t,\bar{x}}(u(.))) \right| = \left| (-y^0)[\, J_{t,x}(u(.)) - J_{t,\bar{x}}(u(.)) \,] \right|$$

$$= \Big| \; (-y^0)[\int_t^T f^o(r, z_{t,x}(r,p), u(r))\, dr + h(z_{t,x}(T,p))$$

$$- \int_t^T f^o(r, z_{t,\bar{x}}(r,p), u(r))\, dr - h(z_{t,\bar{x}}(T,p))] \; \Big|$$

$$\leq \left| -y^0 \right| \int_t^T \left| f^o(r, z_{t,x}(r,p), u(r)) - f^o(r, z_{t,\bar{x}}(r,p), u(r)) \right| dr + \left| -y^0 \right|$$

$$\left| h(z_{t,x}(T,p)) - h(z_{t,\bar{x}}(T,p)) \right|$$

$$\leq \left| -y^0 \right| \int_t^T W(\left| z_{t,x}(r,p) - z_{t,\bar{x}}(r,p) \right|, \left| z_{t,x}(r,p) \right| \vee \left| z_{t,\bar{x}}(r,p) \right|) dr$$

$$+ \left| -y^0 \right| W(\left| z_{t,x}(T,p) - z_{t,\bar{x}}(T,p) \right|, \left| z_{t,x}(T,p) \right| \vee \left| z_{t,\bar{x}}(T,p) \right|)$$

$$\leq \left| -y^0 \right| \int_t^T W(C|x-\bar{x}|, C(1+|x| \vee |\bar{x}|))\, dr + \left| -y^0 \right| W(C|x-\bar{x}|, C(1+|x| \vee |\bar{x}|))$$

$$\leq \left| -y^0 \right| (T+1)\, W(C|x-\bar{x}|, C(1+|x| \vee |\bar{x}|)).$$

Thus by taking the infimum of last inequality over admissible pairs $z(\tau)$, $u(\tau)$, $\tau \in [t,T]$, whose trajectories start at $(t, z(t,p)) \in K$, we obtain (4.21).

Next, we let $0 \leq t \leq \bar{t} \leq T$, $x \in X$, $y^0 < 0$, by (4.12), (4.13), and (4.18) for any $u(.) \in \mathcal{U}[t,T]$, we have

$$\left| -y^0 J_{t,x}(u(.)) - [-y^0 J_{t,x}(\bar{t}, u(.))] \right| = \left| (-y^0)\{ J_{t,x}(u(.)) - J_{t,x}(\bar{t}, u(.)) \} \right|$$

$$\le \left| -y^0 \right| \int_t^{\bar{t}} \{ L + W(|z_{t,x}(r,p)|, |z_{t,x}(r,p)|) \} dr$$

$$+ \left| -y^0 \right| \int_t^{T} W(|z_{t,x}(r,p) - z_{\bar{t},x}(r,p)|, |z_{t,x}(r,p)| \vee |z_{\bar{t},x}(r,p)|) dr$$

$$+ \left| -y^0 \right| W(|z_{t,x}(T,p) - z_{\bar{t},x}(T,p)|, |z_{t,x}(r,p)| \vee |z_{\bar{t},x}(r,p)|)$$

$$\le \left| -y^0 \right| \{ L + W(C(1+|x|), C(1+|x|)) \} (\bar{t} - t)$$

$$+ \left| -y^0 \right| (T+1) W(C \left| (e^{A(\bar{t}-t)} - I) x \right| + C(1+|x|)(\bar{t}-t), C(1+|x|)).$$

Then we can define $\hat{W}$ such that (4.22) holds.

To prove (4.23), let $t \in [0,T]$, and $x \in X$, $y^0 < 0$ for any $u(.) \in \mathcal{U}[t,T]$ by (4.19)

$$\left| -y^0 J_{t,x}(u(.)) - (-y^0 J_{\bar{t}, e^{A(\bar{t}-t)}x}(u(.))) \right| = \left| (-y^0)[J_{t,x}(u(.)) - J_{\bar{t}, e^{A(\bar{t}-t)}x}(u(.))] \right|$$

$$= \Big| (-y^0)[\int_t^T f^o(r, z_{t,x}(r,p), u(r)) dr + h(z_{t,x}(T,p))$$

$$- \int_{\bar{t}}^T f^o(r, z_{\bar{t}, e^{A(\bar{t}-t)}x}(r,p), u(r)) dr - h(z_{\bar{t}, e^{A(\bar{t}-t)}x}(T,p))] \Big|$$

$$\le \left| (-y^0) \int_t^{\bar{t}} (f^o(r, z_{t,x}(r,p), u(r)) - f^o(r, 0, u(r)) + f^o(r, 0, u(r))) dr \right|$$

$$+ \left| (-y^0) \int_{\bar{t}}^T f^o(r, z_{t,x}(r,p), u(r)) - f^o(r, z_{\bar{t}, e^{A(\bar{t}-t)}x}(r,p), u(r)) dr \right|$$

$$+ \left| (-y^0)[h(z_{t,x}(T,p)) - h(z_{\bar{t}, e^{A(\bar{t}-t)}x}(T,p))] \right|$$

$$\le \left| -y^0 \right| \int_t^{\bar{t}} \{ L + W(|z_{t,x}(r,p)|, |z_{t,x}(r,p)|) \} dr$$

$$+ \left| -y^0 \right| \int_{\bar{t}}^T W(|z_{t,x}(r,p) - z_{\bar{t}, e^{A(\bar{t}-t)}x}(r,p)|, |z_{t,x}(r,p)| \vee |z_{\bar{t}, e^{A(\bar{t}-t)}x}(r,p)|) \} dr$$

$$+ \left| -y^0 \right| W(|z_{t,x}(T,p) - z_{\bar{t}, e^{A(\bar{t}-t)}x}(T,p)|, |z_{t,x}(T,p)| \vee |z_{\bar{t}, e^{A(\bar{t}-t)}x}(T,p)|)$$

$$\le \left| -y^0 \right| \{ L + W(C(1+|x|), C(1+|x|)) \} (\bar{t} - t)$$

$$+ \left| -y^0 \right| \{ (T+1) W(C |z_{t,x}(\bar{t},p) - e^{A(\bar{t}-t)}|, C(1+|x|)) \}$$

$$\le \left| -y^0 \right| \{ L + W(C(1+|x|), C(1+|x|)) \} (\bar{t} - t)$$

$$+ \left| -y^0 \right| \{ (T+1) W(C(1+|x|)(\bar{t}-t), C(1+|x|)) \}.$$

Hence we can define $\hat{W}$ such that (4.23) holds. Finally for any $t \in [0,T]$

and $z(0, p(0)) = x \in X$, $y^0 < 0$, $u(.) \in \mathcal{U}[t,T]$ and by (4.12), (4.13)

$$\left| -y^0 J_{t,x}(u(0)) - [-y^0 h(e^{A(T-t)} x)] \right| = \left| (-y^0)[J_{t,x}(u(.)) - h(e^{A(T-t)}x)] \right|$$

$$\leq \left| (-y^0)[\int_t^T f^o(r, z_{t,x}(r,p), u(r))\,dr + h(z_{t,x}(t,p)) - h(e^{A(T-t)}x)] \right|$$

$$\leq \int_t^T \left| (-y^0)[f^o(r, z_{t,x}(r,p), u(r)) - f^o(r, 0, u(r)) + f^o(r, 0, u(r))] \right| dr$$

$$+ \left| -y^0 \right| W(\left| z_{t,x}(T,p) - e^{A(T-t)} x \right|, \left| z_{t,x}(T,p) \right| \vee \left| e^{A(T-t)} \right|)$$

$$\leq \left| -y^0 \right| T\{L + W(C(1+|x|), C(1+|x|))\} + \left| -y^0 \right| W(C(1+|x|)(T-t), C(1+|x|)).$$

Hence we can define $\hat{W}$ such that (4.24) holds. The conclusion under (A_1), (A_2)′, and (A_3)′ can be proved similarly. □

A Verification Theorem:

We will find a dual optimal state feedback control via dual value function, in some generalized sense. Now, let us introduce the following notion. Let *K*, *P* and a function $x(t, p)$ defined on *P* such that $(t, x(t, p)) \in K$, $(t, p) \in P$, be as defined in subsection 4.1. For the function $\varphi: [0,T] \times (-\infty,0] \times X \to \mathbb{R}$, with fixed t_0, $\varphi(t_0, z(0, p(0)))$, $(t, p) \in P \subset \mathbb{R}^2 \times X$ is a function of $z(0, p(0))$ we may define its super differential in $z(0, p(0))$ denoted by $D_x^+\varphi(t_0, z(0, p(0)))$, on the other hand, we define $D^+\varphi(t_0, z(0, p(0)))$ in the following way:

$$D^+\varphi\left(t_0, z(0, p(0))\right) = \left\{ (\alpha, a) \in \mathbb{R}^2 \times X^* \;\middle|\; \overline{\lim_{t\downarrow t_0, z(t,p)\to z(0,P(0))}} \frac{\varphi(t, z(t, p(t))) - \varphi(t_0, z(0, p(0))) - \alpha(t - t_0) - \langle a, z(t, p(t)) - z(0, p(0))\rangle}{|t - t_0| + |z(t, p(t)) - z(0, p(0))|} \leq 0 \right\}$$

Next, in the following Theorem we give a sufficient condition for an admissible pair to be optimal.

Theorem 4.3: Suppose that $z(t) = z(t, p(t))$, $(t,p) \in P \subset \mathbb{R}^2 \times X$ where $(t, z(t,p)) \in K \subset [0,T] \times X$. Now, let ($A_1$), ($A_2$)', and ($A_3$)', hold, let $(\bar{z}(., \bar{p}(.)), \bar{u}(.))$ be an admissible pair suppose that

$$\bar{z}(t, \bar{p}) \in D(A), \quad \text{a.e.,} \quad t \in [0,T]; \quad A\bar{z}(., \bar{p}(.)) \in L^1([0,T], X), \qquad (4.29)$$

and there exists a function $\psi(.)$ such that

$$\left(-y^0\langle \psi(t, \bar{p}), A\bar{z}(t, \bar{p}) + f(t, \bar{z}(t, \bar{p}), \bar{u}(t))\rangle + y^0 f^o(t, \bar{z}(t, \bar{p}), \bar{u}(t)), y^0\psi(t, \bar{p})\right)$$

$$\in D^+ V_D(t, \bar{z}(t, \bar{p})) \quad \text{a.e.,} \; t \in [0,T], \; y^0 < 0 \qquad (4.30)$$

Then $(\bar{z}(., \bar{p}(.)), \bar{u}(.))$ is optimal.

Proof : Let $\theta(t) = V_D(t, \bar{z}(t, \bar{p}))$. Then $\theta(.)$ is continuous and for almost all $t \in [0,T]$, we have (see (4.26), (4.27), (4.16), and (4.18))

$$|\theta(s)-\theta(t)| = \left|V_D(s,\bar z(s,\bar p))-V_D(t,\bar z(t,\bar p))\right| \le \left|-y^0\right| C(|s-t|+\left|(e^{A(s-t)}-I)\,\bar z(t,\bar p)\right|)$$

$$\le \left|-y^0\right| C(1+|A\bar z(t,\bar p)|)\ |s-t|,$$

Where C depends on $z(0, p(0)) = x$, the fixed initial state. Thus, $\theta(.)$ is Lipschitz continuous? On the other hand, for almost all $t \in [0, T]$, we have

$$\bar z(t+\delta,\bar p) = \bar z(t,\bar p)+[A\bar z(t,\bar p)+f(t,\bar z(t,\bar p),\bar u(t))]\delta+o(\delta). \tag{4.31}$$

Thus, by (4.30),

$$\theta(t+\delta)-\theta(t) = V_D(t+\delta,\bar z(t+\delta,\bar p))-V_D(t,\bar z(t,\bar p))$$

$$= V_D(t+\delta,\bar z(t,\bar p)+[A\bar z(t,\bar p)+f(t,\bar z(t,\bar p),\bar u(t))]\delta+o(\delta))-V_D(t,\bar z(t,p))$$

$$\le \delta[-y^0\langle\psi(t,\bar p),A\bar z(t,\bar p)+f(t,\bar z(t,\bar p),\bar u(t))\rangle+y^0\ f^o(t,\bar z(t,\bar p),\bar u(t))]$$

$$+\delta\, y^0\langle\psi(t,\bar p),A\bar z(t,\bar p)+f(t,\bar z(t,\bar p),\bar u(t))\rangle+o(\delta)$$

$$\le \delta\ y^0\ f^o(t,\bar z(t,\bar p),\bar u(t))+o(\delta).$$

This together with the Lipschitz continuity of $\theta(.)$, implies that

$$d\theta(t)/dt \le y^0 f^o(t,\bar z(t,\bar p),u(t)),\quad \text{a.e.},\quad t\in[0,T].\ \text{Then by (4.2)}$$

$$V_D(t,\bar p) \ge -y^0\int_t^s f^o(t,\bar z(t,\bar p),\bar u(t))+V_D(s,\bar p) \ge V_D(t,\bar p)\quad \forall\, 0\le t\le s\le T.$$

Hence, the pair $(\bar z(.,\bar p(.)),\bar u(.))$ is optimal. □

Next, for any $(t,z(0,p(0)))=(t,x)\in[0,T]\times D(A)$, we define

$$G(t,x)=\left\{u\in U\ \middle|\ \lim_{\delta\to 0}\frac{V_D(t+\delta,x+\delta[Ax+f(t,x,u)]-V_D(t,x)}{\delta}=y^0 f^o(t,x,u)\right\}.$$

We have the following result.

Theorem 4.4: Suppose that $z(t)=z(t,p),(t,p)\in P\subset\mathbb{R}^2\times X$, where $(t,z(t,p))\in K\subset[0,T]\times X$, and let ($A_1$), ($A_2$)', and ($A_3$)' hold. Suppose that $(\bar z(.,p(.)),\ \bar u(.))$ be an admissible pair, such that (4.29) holds. Then the following are equivalent:

(i) $(\bar z(.,\bar p(.)),\bar u(.))$ is optimal ;

(ii) It holds that

$$\bar u(t)\in G(t,\bar z(t,\bar p))\quad \text{a.e.},\ t\in[0,T]. \tag{4.32}$$

Proof: Because $\bar z(t,\bar p)\in D(A)$, a.e., $t\in[0,T]$, by proof of Th. 4.3, we know that $\theta(t)=V_D(t,\bar z(t,\bar p))$ is Lipschitz continuous.

(i) $\Rightarrow$(ii). Let $(\bar z(.,\bar p(.)),\bar u(.))$ be optimal. By (4.31),

$$\lim_{\delta\to 0}\frac{V_D(t+\delta,\bar z(t,\bar p)+\delta[A\bar z(t,\bar p)+f(t,\bar z(t,\bar p),\bar u(t))]-V_D(t,\bar z(t,\bar p))}{\delta}$$

$$=\lim_{\delta\to 0}\frac{V_D(t+\delta,\bar z(t,\bar p)+o(\delta))-V_D(t,\bar z(t,\bar p))}{\delta}$$

$$=\lim_{\delta\to 0}\frac{1}{\delta}\,y^0\int_t^{t+\delta} f^o(r,\bar z(r,\bar p),\bar u(r))\,dr = y^0 f^o(t,\bar z(t,\bar p),\bar u(t))\quad \text{a.e.},\ t\in[0,T],\ y^0<0.$$

This means that (4.32) holds.

(ii) $\Rightarrow$(i). In this case, we have

$$\lim_{\delta \to 0} \frac{V_D(t+\delta, \bar{z}(t,\bar{p})) - V_D(t, \bar{z}(t,\bar{p}))}{\delta}$$
$$= \lim_{\delta \to 0} \frac{V_D(t+\delta, \bar{z}(t,\bar{p}) + \delta[A\bar{z}(t,\bar{p}) + f(t,\bar{z}(t,\bar{p}),\bar{u}(t))] + o(\delta)) - V_D(t,\bar{z}(t,\bar{p}))}{\delta}$$
$$= y^0 f^o(t, \bar{z}(t,\bar{p}), \bar{u}(t)), y^0 < 0.$$

Then, by the absolute continuity of $V_D(t, \bar{z}(t,\bar{p}))$ we obtain the optimality of $\bar{u}$ (.) . □

We see that (4.32) gives a representation of the optimal control in terms of the corresponding optimal state trajectory. Such a form is referred to as a state feedback control. Formally, the dual optimal trajectory $\bar{z}(t,\bar{p})$ satisfies the following:

$$d/dt\, \bar{z}(t,\bar{p}(t)) \in A\bar{z}(t,\bar{p}(t)) + f(t,\bar{z}(t,\bar{p}(t)), G(t,\bar{z}(t,\bar{p}(t)))),\ t \in [0, T]. \tag{4.33}$$

This is a ***differential inclusion*** in the unknown function $\bar{z}(t,\bar{p}(t))$ and the control variable does not appear explicitly. Such a system is called the closed –loop system for our Problem(C). Roughly speaking, in order to solve Problem(C), we first find a solution of (4.33). We then use (4.32) to determine an optimal control. We refer to Th. 4.4, as an optimal synthesis.

Example:

Let X be a Hilbert space of infinite dimension. Let $A = A^*: D(A) \subset X \to X$ be a self–adjoint operator with the following properties: There exist sequences $\{\varphi_k\}_{k\geq 1} \subset X$ and $\{\lambda_k\}_{k\geq 1} \subset \mathbb{R}$ with the properties that $\{\varphi_k\}_{k\geq 1}$ forms an orthonormal basis of X and $\{e^{\lambda k}\}_{k\geq 1}$ forms a basis of $L^2(0,1)$, such that

$0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lim_{k\to\infty} \lambda_k = +\infty,$

$A\varphi_k = -\lambda_k \varphi_k,\ k \geq 1.$

This can be easily achieved [9]. Now, we suppose that

$$b \in X,\ b_k \equiv \langle b, \varphi_k \rangle \neq 0, k \geq 1 \tag{4.34}$$

$$a \in X,\ c \in \mathbb{R},\ c \neq \langle a, A^{-1}a \rangle \tag{4.35}$$

$$q = (e^A - I)A^{-1}b\ ;\ U = [-2,2], \tag{4.36}$$

Also let $f(t, z(t), u(t)) \equiv b\,u(t)$, $t \in [0,1]$, and consider the following system:

$$\dot{z}(t) = Az(t) + bu(t),\ t \in [0,1], \tag{4.37}$$

With $u(.) \in \mathrm{U} \equiv \{ u(.): [0,1] \to U \mid u(.) \text{ measurable}\}$.

Our constraint for the endpoints of the state $z(0) = 0$ and $z(1) = q$, thus $(z(0),z(1)) \in \{0\}\times\{q\}$ (closed and convex) $\subset X \times X$.

Now, let $f^o(t,z(t),u(t)) \equiv \langle a, z_{t,z(0)}(t)\rangle + cu(t),\ t \in [0,1]$, and the cost functional is given by

$$J_{t,z(0)}(u(.)) = \int_0^1 [\langle a, z_{t,z(0)}(t)\rangle + cu(t)]dt + h_{t,z(0)}(z(1)). \tag{4.38}$$

Then it is not hard to see that the dual value function is given by

$$V_D(t,z(t,p)) = \inf_{u(.)\in \mathcal{U}} (-y^0 J_{t,z(0)}(u(.))),\quad y^0 \leq 0. \tag{4.39}$$

Solutions to system (4.37) are understood to be mild solution [20]. Thus, for given $u(.) \in \mathrm{U}$ and initial state $z(0, p(0)) = 0$, we have

$$z(t,p) = \int_0^t e^{A(t-s)} bu(s)ds,\ t \in [0,1]. \tag{4.40}$$

Here, we suppose that $(\bar{z}(.,.), \bar{u}(.))$ be an admissible pair for our problem, and let $\psi(.)$ be the solution of the following equation

$$\psi(t) = -e^{A^*(1-t)} h_x(\bar{z}(1,\bar{p})) + \int_t^1 e^{A^*(r-t)} f_x(r,\bar{z}(r,\bar{p}),\bar{u}(r))^* \psi(r)dr$$

$$- \int_t^1 e^{A^*(r-t)} f_x^o(r,\bar{z}(r,\bar{p}),\bar{u}(r))dr, \; t \in [0,1], \tag{4.41}$$

With terminal condition $\psi(1) = y^0 h_x(\bar{z}(1,\bar{p})), y^0 \le 0.$

For any admissible pair (z(.,.), u(.)), also let us define the following

$$\mathbf{T}\,(z(.,.), u(.)) \equiv \{\, t \in [0,1] \,|\, z(t, p(t)) \in D(A),$$
$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \int_t^{t+\delta} |\; f(r,z(r,p),u(r)) - f(t,z(t,p),u(t)) \;| dr = 0\}.$$

Now, we fix a $t \in \mathbf{T}\;(\bar{z}(.,\bar{p}),\bar{u}(.))$, then for any $s \in X$ and $\tau \in (t, 1]$, it is not difficult to see that [21]:

$$V_D(\tau,s) - V_D(t,\bar{z}(t,\bar{p})) \le -y^0 \int_\tau^1 f^o(r,z_{\tau,s}(r,p),\bar{u}(r))dr +$$

$$y^0 \int_t^1 f^o(r,\bar{z}\;\;(r,\bar{p}),\bar{u}(r))dr + y^0 h(z_{\tau,s}(1,p)) - y^0 h(\bar{z}(1,\bar{p}))$$

$$= -y^0 \langle \psi(t), s - \bar{z}(t,\bar{p}) \rangle + (\tau - t)[-y^0 \langle \psi(t,p), A\bar{z}(t,\bar{p}) + f(t,\bar{z}(t,\bar{p}),\bar{u}(t)) \rangle$$
$$+ y^0 f(t,\bar{z}(t,\bar{p}),\bar{u}(t))] + o(|\tau - t| + |s - \bar{z}(t,\bar{p})|). \tag{4.42}$$

Here, if we take $\bar{u}(.) \equiv 1$, $t \in [0,1]$, then by (4.36), and (4.40) the corresponding trajectory denoted by $\bar{z}(.) = \bar{z}(.,\bar{u}(.))$ satisfies $z(1) = \int_0^1 e^{As} b\, ds = (e^A - I)A^{-1}b = q.$

Thus, it is not difficult to check that, for $\bar{u} \equiv 1$ we can obtained from (4.42) the following

$$(-y^0 \langle \psi(t,\bar{p}), A\bar{z}(t,\bar{p}) + f(t,\bar{z}(t,\bar{p}),1) \rangle + \;\; y^0 f(t,\bar{z}(t,\bar{p}),1), y^0\, \psi(t,\bar{p})) \in D^+ V_D(t,\bar{z}(t,\bar{p})$$

Therefore, by using Theorem 4.3, we see that the control $\bar{u} \equiv 1$ $t \in [0,1]$ is the optimal control for our problem.

Conclusions and future work:

Conclusions:

1. Nonclassical approach to DP via semigroup theory for optimal control problem (C) is described (section 4.2).

2. Dual value function V_D(., .) (4.2), for Problem (C) is defined and proved it a solution to the (DHJB) equation (4.4), which is essential in the study of optimality.

3. Some properties of V_D(., .) have been presented, such as, various kinds of continuities and boundedness, these properties used to give a sufficient condition for optimality (Theorem 4.3). Also

4. A verification theorem to find an optimal state feedback control via V_D(., .) is proved (Theorem 4.4).

Future Work:

1. Semigroup for differential inclusion may be considered.

2. V_D(., .) is a viscosity solution of (DHJB) equation may be proved.

Another kind of continuity of V_D(., .), such as, B–continuity which is essential in the study of optimality may be studied. Also, another interesting property of V_D(., .) (Semiconcavity) may be established.

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