DOI: 10.24996/ijs.2024.65.9.29

ISSN: 0067-2904

Pure Rickart Modules

Hassan Sabti Al-rdeny[1], Bahar Hamad Al-Bahrani[2]
Department of mathematic, College of Science, University of Baghdad, Baghdad, Iraq

Received: 13/4/2023 Accepted: 1/8/2023 Published: 30/9/2024

Abstract
This paper focuses on the study of pure Rickart modules (or PR-modules for short), which are a class of modules over a commutative ring with identity. The main objective is to investigate the properties and characterizations of these modules, as well as their relationship with other classes of modules such as free, projective, and flat modules. This paper also explores the connections between PR-modules and various algebraic structures such as rings. The results obtained in this study provide a deeper understanding of the structure and behavior of PR-modules, which can have important applications in algebraic geometry, representation theory, and other areas of mathematics. Some results about PR-module have been investigated in this paper, for example we demonstrate a module M is PR-module if and only if for every $g \in End(M)$, $C_M \cap T_g$ is pure submodule of $M \oplus M$ (or $C_M \cap T_g \leq_P M \oplus M$ for short). Also, some kind of generalization of these rings have been constructed and demonstrated in term of PR-modules.

Keywords: Pure Rickart modules, Direct summand, Kernel of endomorphism, pure submodules, flat modules.

مقاسات ريكارتية النقية

حسن سبتي الرديني ، بهار حمد البحراني
قسم الرياضيات ،كلية العلوم ,جامعة بغداد ,بغداد ,العراق

الخلاصة
يركز هذا البحث على دراسة مقاسات ريكارتيه نقية (أو مقاسات-PR باختصار) ، وهي فئة من المقاسات المعرفة على حلقة تبادلية ذات عنصر محايد. الهدف الرئيسي هو التحقيق في خصائص وتوصيفات هذه المقاسات ، بالإضافة إلى علاقتها بفئات أخرى من المقاسات مثل المقاسات الحرة والإسقاطية والمسطحة. يستكشف البحث أيضًا الروابط بين المقاسات الريكارتيه النقية والتراكيب الجبرية المختلفة مثل الحلقات. توفر النتائج التي تم الحصول عليها في هذه الدراسة فهمًا أعمق لبنية وسلوك المقاسات ريكارتيه نقية، والتي يمكن أن يكون لها تطبيقات مهمة في الهندسة الجبرية ، ونظرية التمثيل ومجالات أخرى من الرياضيات. تم التحقيق في هذه الورقة في بعض النتائج حول مقاسات-PR ، على سبيل المثال نوضح أن المقاس M هو مقاس-PR إذا وفقط إذا كان لكل $g \in End(M)$ فان $C_M \cap T_g \leq_P M \oplus M$. أيضًا تم إنشاء نوع من الاعمام هذه الحلقات وإثباتها من خلال مقاسات-PR.

*Email: hassanalrdeny308@gmail.com

1. INTRODUCTION

PR-modules have been extensively studied in the field of module theory and have many interesting properties. They also have applications in other areas of mathematics, such as commutative algebra and algebraic geometry. The study of PR-modules has led to the development of several important concepts and techniques in module theory, including the use of torsion theories to classify certain classes of modules. The paper on PR-modules has focused on several different areas, including their structure and classification, their connections to other classes of modules, and their applications in various fields. We introduce the definition for PR-module as follows; if for every $f \in End(M),\ then\ Ker\ f\ \leq_P M$, where M is a module. In particular, if $M = R$, then R is called PR-ring if R is pure Rickart as R-module. In the other side, PR-ring can be obtained from $ann(a)$, $a \in R$ is pure ideal of R, see [1] . Additionally, if we have two modules say M_1, M_2 are R-modules, then M_1 is an M_2-PR-module (or relatively PR-module to M_2), if satisfy the following condition, for every R-homomorphism $f: M_1 \to M_2$, $\ker f \leq_P M_1$, see [1]. Recall that an R-module M is called a prime R-module if $ann(x) = ann(y)$, for every non-zero elements $x\ and\ y$ in M, see [2]. Let us recall that a ring R is called a Bezout ring if every finitely generated ideal is principal, see [3]. Recall that an R-module M is called a Quasi Dedekind R-module if every non-zero endomorphism of M is a monomorphisem, see [4]. Recall that a ring R is a pure simple if 0 and R are the only pure ideals of R, see [5]. Recall that a ring R is a PF-ring if every principal ideal is a flat ideal in R, see [6]. Recall that a ring R is a flat ring if every finitely generated ideal in R is flat, equivalently, every ideal in R is flat, see [7]. Recall that an R-module M is called a flat module, if for every short exact sequence of R-module: $0 \to A \to B \to C \to 0$ the sequence $0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0$ is also exact, see [8]. Let M be R-module. Recall that $Z(M) = \{x \in M : ann(x) \leq_e R\}$ is called singular submodule of M. If $Z(M) = M$, then M is called the singular module. If $Z(M) = 0$, then M is called the nonsingular module, see [9]. Recall that a submodule N of an R-module M is called a fully invariant submodule if for every endomorphism $f: M \to M, f(N) \subseteq N$, see [10].

In this article, we provide some findings on the PR-modules.
In section **2**, we provide a description of PR-modules. We also research relation between flat and PR-modules. For instance, we demonstrate that, if M is flat R-module, then M is a PR-module if and only if for every R-homomorphism $g: M \to M$, $C_M + T_g$ is flat, see Corollary 2.18.

In Section **3**, we characterize specific ring classes in terms of the PR-modules. For instance, we illustrate that a ring R is flat if and only if every projective R-modules are relatively PR-module to any flat R-module, see Theorem 3.13.
Everywhere else in this article, R represent ring with identity and M is a unital left R-module. For a left module M, $End(M)$ that will mean the endomorphism ring of M. The observes $K \leq M$, $K \leq_P M$ mean that K is a submodule, a pure submodule of M.

2. Pure Rickart Modules by Means of Flat Module

This section provides a characterization for the PR-modules by means of flat module. We illustrate that a flat module need not be a PR-module and the converse is not true in general, see Remark 2.5.

Proposition 2.1: Let $M_1$and M_2 be R-modules such that $\forall g \in Hom(M_1, M_2), Im\ g$ is flat, then M_1 is M_2-PR-module.
Proof: Let $g: M_1 \to M_2$ be an R-homomorphism. We want to show that $\ker g \leq_P M_1$. Consider the short exact sequence

$$0 \to \ker g \xrightarrow{i} M_1 \xrightarrow{g} Im\ g \to 0.$$

Since $Im\ g$ is flat, therefore, $\ker g \leq_P M_1$, by [8, Proposition 3.67, p. 147].

Proposition 2.2: Let M_1 be a flat R-module and M_2 be an R-module, then M_1 is M_2-PR-module if and only if for every $g: M_1 \to M_2$ be an R-homomorphism, $Im\ g$ is flat.

Proof: Let M_1 is M_2-PR-module and let $g: M_1 \to M_2$ be an R-homomorphism. We want to show that $Im\ g$ is flat. Consider the shirt exact sequence

$$0 \to \ker g \xrightarrow{i} M_1 \xrightarrow{g} Im\ g \to 0$$

Since M_1 is M_2-PR-module, therefore, $\ker g \leq_P M_1$. Since M_1 is flat by our assumposion. Thus $Im\ g$ is flat, by [8, Proposition 3.60, p.139].

The converse follows by Proposition 2.1.

Corollary 2.3: Let M be a free (projective) R-module, then M is PR-module if and only if $\forall g \in Hom(M, M)$, $Im\ g$ is flat.

Proof: Let M be a free (projective) R-module and hence M is flat and let $g: M \to M$ be an R-homomorphism. By Proposition 2.2, the result follows.

The converse follows by Proposition 2.1.

Corollary 2.4: Let R be a pure simple ring. If R is PR-ring, then R is integral domain (ID).

Proof: Let Ra be a principle ideal in R. To show that Ra is flat. Let $g: R \to Ra$ be a map define by $g(r) = ra, \forall r \in R$. It's clear that g epimorphism. Consider the short exact sequence

$$0 \to \ker g \xrightarrow{i} R \xrightarrow{g} Ra \to 0$$

Since R is PR-ring, then Ra is flat, by Corollary 2.3 and since R is pure simple and PF-ring. Thus R is ID, by [5].

Remark 2.5: A flat module need not be a PR-module. Also, the converse is not true. For example, the module Z_4 as Z_4-module. Since Z_4 is free, therefore Z_4 is flat Z_4-module. Now define a map $g: Z_4 \to Z_4$ by $g(x) = 2x, \ \forall x \in Z_4$. Since $\ker g = \{\bar{0}, \bar{2}\}$ is not pure in Z_4, therefore, Z_4 is not PR-module. For the converse, the module Z_6 as Z-module. Since $6Z_6 = 0$, then Z_6 is not torsion free. Therefore, Z_6 is not flat. But Z_6 is semisimple, so Z_6 is PR-module.

Proposition 2.6: Let R a Bezout domain and let M_2 be a torsion free (flat, projective) R-module, then every R-module M_1 is M_2-PR-module.

Proof: Let M_1 be an R-module and let $g: M_1 \to M_2$ be an R-homomorphism. Consider the short exact sequence $0 \to \ker g \xrightarrow{i} M_1 \xrightarrow{g} Im\ g \to 0$

Since M_2 is a torsion free and $Im\ g \leq M_2$, therefore $Im\ g$ is torsion free. But R is Bezout domain, then $Im\ g$ is flat, by [3, Corollary 2.2. 3.1, p.23]. Therefore $\ker g \leq_P M_1$, by [11]. Thus M_1 is M_2-PR-module.

Proposition 2.7: Let R a Bezout domain and let M_2 be a nonsingular R-module, then every R-module M_1 is M_2-PR-module.

Proof: Let M_1 be an R-module and let $g: M_1 \to M_2$ be an R-homomorphism. Consider the short exact sequence $0 \to \ker g \overset{i}{\to} M_1 \overset{g}{\to} Im\ g \to 0.$
Since R is ID, therefore, $T(M_2) = Z(M_2) = 0$ and hence M_2 is torsion free. But $Im\ g \leq M_2$, then $Im\ g$ is torsion free. Since R is Bezout domain, therefore $Im\ g$ is flat, by [3, Corollary 2.2. 3.1, p.23]. Hence $\ker g \leq_P M_1$. Thus M_1 is M_2-PR-module.

Proposition 2.8: Let R be ID and let M_2 be a singular R-module. Then for every flat R-module M_1, either $Hom(M_1, M_2) = 0$ or M_1 is not M_2-PR-module.

Proof: Let $Hom(M_1, M_2) \neq 0$. To show that M_1 is not M_2-PR-module. By contradiction assume that M_1 is M_2-PR-module and let $g: M_1 \to M_2$ be an R-homomorphism, then $\ker g \leq_P M_1$. Consider the short exact sequence

$$0 \to \ker g \overset{i}{\to} M_1 \overset{g}{\to} Im\ g \to 0$$

Hence $Im\ g$ is flat. Since R be ID, then $Im\ g$ is torsion free, by [8, Proposition 3.49, p.134]. By the first isomorphism theorem $\frac{M_1}{\ker g} \cong Im\ g$, so $\frac{M_1}{\ker g}$ is torsion free. Since M_2 be a singular R-module, then $T(M_2) = Z(M_2) = M_2$ and hence M_2 is torsion. But $Im\ g \leq M_2$, then $Im\ g$ is torsion. Therefore, $Im\ g = 0$ which is a contradiction. Thus M_1 is not M_2-PR-module.

Remark 2.9: Let M be an R-module and $g: M \to M$ be an R-homomorphisem.
Let $C_M = M \oplus 0, D_M = 0 \oplus M$ and $\bar{g}: C_M \to D_M$ be a map define by $\bar{g}(m, 0) = (0, g(m))$, for every $m \in M$. It is clear that $M \oplus M = C_M \oplus D_M$, $\bar{g}$ is an R-homomorphism and $\ker \bar{g} = \ker g \oplus 0$. Let $T_g = \{x + \bar{g}(x), x \in A_M\}$. Clearly that $T_g \leq M \oplus M$ and $M \oplus M = T_g \oplus D_M$.
In this article by $C_M, D_M, \overline{g},\ T_g$ we mean the same concept as stated in the remark above, [12].

Theorem 2.10: Let $M_1\ and\ M_2$ be two R-modules. then M_1 is M_2-PR-module if and only if for every R-homomorphism $g: M_1 \to M_2$, $C_{M_1} \cap T_g \leq_P M_1 \oplus M_2$.

Proof: Let M_1 is M_2-PR-module and let $g: M_1 \to M_2$ be an R-homomorphism. Since M_1 is M_2-PR-module, then $\ker g \leq_P M_1$ and hence $\ker \bar{g} = \ker g \oplus 0$. Therefore, $\ker \bar{g} \leq_P M_1 \oplus M_2$. Claim that $\ker \bar{g} = C_{M_1} \cap T_g$. By the same argument of the proof of the, [13]. To show that, let $(m, 0) \in \ker \bar{g}$, then $\bar{g}(m, 0) = (0,0)$, where $m \in M$. Hence $(m, 0) = (m, 0) + \bar{g}(m, 0) \in C_{M_1} \cap T_g$. Now, let $(m, 0) \in C_{M_1} \cap T_g$, so there exists $m_1 \in M$ such that $(m, 0) = (m_1, 0) + \bar{g}(m_1, 0) = (m_1, 0)(0, g(m_1))$. Since $(0, g(m_1)) \in C_{M_1} \cap D_{M_2} = 0$, then $g(m_1) = 0$. Hence, $m = m_1$ and $g(m) = g(m_1) = 0$. Therefore, $(m, 0) \in \ker \bar{g}$. Thus $\ker \bar{g} = C_{M_1} \cap T_g \leq_P M_1 \oplus M_2$.
Conversely, let $g: M_1 \to M_2$ be an R-homomorphism. Since $C_{M_1} \cap T_g = ker\ \bar{g} \leq_P M_1 \oplus M_2$ and $\ker \bar{g} \leq C_{M_1}$, then $\ker \bar{g} = \ker g \oplus 0 \leq_P M_1 \oplus 0$. Therefore $\ker g \leq_P M_1$. Thus M_1 is M_2-PR-module .

Corollary 2.11: An R-module M is PR-module if and only if for every R-homomorphism $g: M \to M$, $C_M \cap T_g \leq_P M \oplus M$.

Proof: Follows from Theorem 2.10, take $M = M_1 = M_2$.

Theorem 2.12: Let M_1 *and* M_2 be two R-modules. Then M_1 is M_2-PR-module if and only if $I(C_{M_1} \cap T_g) = IC_{M_1} \cap IT_g$, for every R-homomorphism $g: M_1 \to M_2$ and finitely generated ideal I of R.

Proof: Assume that M_1 is M_2-PR-module, then $C_{M_1} \cap T_g \leq_P M_1 \oplus M_2$. Let $g: M_1 \to M_2$ be an R-homomorphism and I be a f.g ideal of R. Hence $I(C_{M_1} \cap T_g) = I(M_1 \oplus M_2) \cap (C_{M_1} \cap T_g)$. It is clear that $I(C_{M_1} \cap T_g) \subseteq IC_{M_1} \cap IT_g$. But $IC_{M_1} \cap IT_g \subseteq \left(I(M_1 \oplus M_2) \cap C_{M_1}\right) \cap T_g = I(M_1 \oplus M_2) \cap \left(C_{M_1} \cap T_g\right) = I\left(C_{M_1} \cap T_g\right)$. Thus $IC_{M_1} \cap IT_g = I(C_{M_1} \cap T_g)$.
Conversely, let $g: M_1 \to M_2$ be an R-homomorphism and let I be a f.g of R. Then $I(M_1 \oplus M_2) \cap (C_{M_1} \cap T_g) = (I(M_1 \oplus M_2) \cap C_{M_1}) \cap T_g = IC_{M_1} \cap T_g$. Similarly,
$I(M_1 \oplus M_2) \cap (C_{M_1} \cap T_g) = (I(M_1 \oplus M_2) \cap T_g) \cap C_{M_1} = IT_g \cap C_{M_1}$
because $C_{M_1}, T_g \leq_P M_1 \oplus M_2$. Therefore, $I(M_1 \oplus M_2) \cap \left(C_{M_1} \cap T_g\right) \subseteq IC_{M_1} \cap IT_g = I(C_{M_1} \cap T_g)$. So, $C_{M_1} \cap T_g \leq_P M_1 \oplus M_2$. Thus M_1 is M_2-PR-module, by Theorem 2.10.

Corollary 2.13: Let M be an R-module, then M is a PR-module if and only if $I\left(C_M \cap T_g\right) = IC_M \cap IT_g$, for every R-homomorphism $g: M \to M$ and finitely generated ideal I of R.

Proof: Follows from Theorem 2.12, take $M = M_1 = M_2$.

Proposition 2.14: Every prime R-module M over a Bezout domain is a PR-module.

Proof: Let $g: M \to M$ be an R-homomorphism and let I be a finitely generated ideal of R. Since R is a Bezout domain, then $I = (r)$ for some $r \in R$. We want to show that $r(C_M \cap T_g) = rC_M \cap rT_g$. Let $0 \neq x \in rC_M \cap rT_g$, hence $x = ra = rb$, $a \in C_M$ and $b \in T_g$. So $r(a - b) = 0$. Assume $a \neq b$. Since $r \in ann(a - b)$ and M is prime, then $r \in ann\,(a)$ *and* $x = 0$ which is a contradiction. Thus $a = b$ *and* $x \in r(C_M \cap T_g)$. So by Corollary 2.13, M is PR-module.

Proposition 2.15: Let M_1 and M_2 be an R-modules such that for every R-homomorphism $g: M_1 \to M_2$, $C_{M_1} + T_g$ is flat, then M_1 is an M_2-PR-module.

Proof: Let $g: M_1 \to M_2$ be an R-homomorphism. Consider the following short exact sequences

$$0 \to C_{M_1} \cap T_g \overset{i_1}{\to} C_{M_1} \overset{\pi_1}{\to} \frac{C_{M_1}}{C_{M_1} \cap T_g} \to 0.$$

$$0 \to T_g \overset{i_2}{\to} C_{M_1} + T_g \overset{\pi_2}{\to} \frac{C_{M_1} + T_g}{T_g} \to 0.$$

Where i_1, i_2 are the inclusion homomorphisms and π_1, π_2 are the natural epimorphisms. By the second isomorphism theorem $\frac{C_{M_1}}{C_{M_1} \cap T_g} \cong \frac{C_{M_1} + T_g}{T_g}$. Since $T_g \leq_P M_1 \oplus M_2$ and hence $T_g \leq_P C_{M_1} + T_g$, by [14]. But $C_{M_1} + T_g$ is flat. Therefore, by [8, Proposition 3.60, p.139], $\frac{C_{M_1}}{C_{M_1} \cap T_g} \cong \frac{C_{M_1} + T_g}{T_g}$ is flat. Thus $C_{M_1} \cap T_g \leq_P C_{M_1}$ by [11]. But $C_{M_1} \leq_P M_1 \oplus M_2$, therefore $C_{M_1} \cap T_g \leq_P M_1 \oplus M_2$ by [14]. Hence M_1 is an M_2-PR-module, by Theorem 2.10.

Corollary 2.16: Let M be an R-module such that for every R-homomorphism $g: M \to M$, $C_M + T_g$ is flat, then M is a PR-module.

Proof: Follows from Theorem 2.15, take $M = M_1 = M_2$.

Theorem 2.17: Let M_1 and M_2 are flat R-modules, then M_1 is M_2-PR-module if and only if for every R-homomorphism $f: M_1 \to M_2$, $C_{M_1} + T_g$ is flat.

Proof: Suppose that M_1 is M_2-PR-module. Let I be a f.g ideal of R. Consider the following short exact sequence

$$0 \to C_{M_1} \cap T_g \overset{f_1}{\to} C_M \oplus T_g \overset{g_1}{\to} C_{M_1} + T_g \to 0,$$

where $f_1(x) = (x, -x), for\ each\ x \in C_{M_1} \cap T_g and\ g_1(a, b) = a + b, for\ each\ a \in C_{M_1}\ and\ b \in T_g$.

Now we construct the following diagram

$$\begin{array}{ccccc} I\otimes(C_{M_1} \cap T_g) & \overset{I\otimes f_1}{\longrightarrow} & I\otimes(C_{M_1} \oplus T_g) & \overset{I\otimes g_1}{\longrightarrow} & I\otimes(C_{M_1} + T_g) \to 0 \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\ 0 \to IC_{M_1} \cap IT_g & \underset{\bar{f}_1}{\to} & IC_{M_1} \oplus IT_g & \underset{\bar{g}_1}{\to} & IC_{M_1} + IT_g \to 0, \end{array}$$

where $\overline{f_1}(x) = (x, -x), for\ each\ x \in IC_{M_1} \cap IT_g\ and\ \bar{g}_1(a, b) = a + b,\ for\ each\ a \in IC_{M_1}\ and\ b \in IT_g$.

$\alpha(r \otimes x) = rx,\ for\ each\ r \in I\ and\ x \in C_{M_1} \cap T_g$.

$\beta\big(r \otimes (a, b)\big) = (ra, rb),\ for\ each\ r \in I, a \in C_{M_1}\ and\ b \in T_g$.

$\gamma\big(r \otimes (a + b)\big) = ra + rb,\ for\ each\ r \in I, a \in C_{M_1}\ and\ b \in T_g$.

It is easily checked that the diagram is commutative. Since $C_{M_1}, T_g \leq_P M_1 \oplus M_2$ and M is flat, then by [14] C_{M_1} and T_g are flat and hence $C_{M_1} \oplus T_g$ is flat. By [8, proposition 2.58, p.81] $I\otimes(C_{M_1} \oplus T_g) \cong I(C_{M_1} \oplus T_g)$= $IC_{M_1} \oplus IT_g$. Thus β is an isomorphism. Therefore α is an epimorphism if and only if γ is monomorphism [8, proposition 2.72, p.90]. It is early show that $\alpha(I\otimes(C_{M_1} \cap T_g))$= $I(C_{M_1} \cap T_g)$. Hence α is onto if and only if M_1 *is* M_2-PR-module, by Theorem (2.12). Moreover, γ is a monomorphism if and only if $I\otimes(C_{M_1} + T_g) \cong \gamma$ $(I\otimes(C_{M_1} + T_g)) = I(C_{M_1} + T_g)$. Thus γ is monomorphism if and only if $C_{M_1} + T_g$ is flat by [8, Proposition 2.58, p.81].

The converse follows from Proposition 2.15.

Corollary 2.18: Let M be a flat R-module, then M is a PR-module if and only if for every R-homomorphism $g: M \to M$, $C_M + T_g$ is flat.

Proof: Follows from Theorem 2.17, take $M = M_1 = M_2$.

3. Characterization of Rings by Means of Pure Rickart Modules

This section study direct summand of PR- modules and provide some kind of generalization of rings have been constructed and demonstrated in term of PR-modules.

Proposition 3.1: Let $M_1 = K_1 \oplus K_2$ and M_2 be an R-modules. If M_1 is M_2-PR-module, then K_1 is M_2-PR-module.

Proof: Suppose that M_1 is M_2-PR-module. Let $g: K_1 \to M_2$ be an R-homomorphism and $P: M_1 \to K_1$ be the projection map. Consider the map $gop: M_1 \to M_2$. Since M_1 is M_2-PR-module, therefore ker $gop \leq_P M_1$. But

$$\begin{aligned} \ker(gop) &= \{x \in M_1;\ gop(x) = 0\} \\ &= \{\alpha + \beta \in M_1;\ g\big(p(\alpha + \beta)\big) = 0, \alpha \in K_1, \beta \in K_2\} \\ &= \{\alpha + \beta \in M_1; g(\alpha) = 0, \alpha \in K_1, \beta \in K_2\} \end{aligned}$$

$= \ker g \oplus K_2$.
Hence $\ker g \leq_P M_1$. But $\ker g \subseteq K_1$, therefore, $\ker g \leq_P K_1$ by [14]. Thus K_1 is PR-module.

Corollary 3.2: Direct summand of PR-module is also PR-module.

Proof: Follows from Proposition 3.1, take $M = M_1 = M_2$.

Proposition 3.3: Let $M = \underset{j\in J}{\oplus} M_j$ be a direct sum of fully invariant submodules $M_j, \forall j \in J$. Then M is PR-module if and only if M_j is PR-module, $\forall j \in J$.

Proof: ⇒) Clear by Corollary 3.2.
Conversely, let $g: M \to M$ be R-homomorphism. To show that $\ker g \leq_P M$. Since M_j be a fully invariant submodule, $\forall j \in J$ then we can consider $g|_{Mj}: M_j \to M_j, \forall j \in J$. Clearly that $\ker g|_{Mj} = \ker g \cap M_j, \forall j \in J$.
Claim that $ker\ g = \oplus_{j\in J} (ker\ g|_{M_j})$. By the same argument of the proof of the, [15].
To show that, let $x \in \ker g$ and let $x = \sum_{j\in J} x_j$, where $x_j \in M_j, for\ each\ j \in J\ and\ x_j \neq 0\ for\ at\ most\ a\ finite\ number\ of\ j \in J$. Hence $g(x) = g\left(\sum_{j\in J} x_j\right) = \sum_{j\in J} g(x_j) = 0$. Thus $g(x_j) = 0, j \in J\ and\ hence\ x_j \in \ker g \cap M_j,\ \forall j \in J$. Therefore, $x \in \oplus_{j\in J} \left(\ker g \cap M_j\right) = \oplus_{j\in J} (\ker g|_{M_j})$. Thus $\ker g = \oplus_{j\in J} (\ker g|_{M_j})$. But M_j be a PR-module, for each $j \in J$. Therefore, $\ker g|_{M_j} \leq_P M_j$ and hence $\ker g \leq_P M$. Thus M is PR-module.

Proposition 3.4: Let M_1 not flat R-module. Then there exists a free R-module F such that F is not M_1-PR-module.

Proof: Assume that M_1 is not flat R-module. Then there exists a free R-module F and an eipmorphism $g: F \to M_1$, by [11, Corollary 4.4.4, p.89]. Claim that $\ker g$ is not pure of F. To illustrate that, assume not. Consider the short exact sequence

$$0 \to \ker g \overset{i}{\to} F \overset{g}{\to} M_1 \to 0,$$

where i is the inclusion map. By [8, Proposition 3.60, p. 139], so M_1 is flat which is contradiction. Therefore, $\ker g$ is not pure of F. Thus F is not M_1-PR-module.

Proposition 3.5: Let M_1 be an R-module. The following conditions are equivalent:
1- M_1 is PR-module;
2- For every $K_1 \leq M_1$, every direct summand K_2 of $M_1$is K_1-PR-module;
3- For every pair of direct summands K_1 and K_2 of M_1 and any $g \in Hom(M_1, K_1)$. The kernel of the restricted map $g|_{K_2}$is a pure of K_2.

Proof: (1) ⇒ (2) Let K_2 be a direct summand of M_1, $K_1 \leq M_1$ and $g_1: K_2 \to K_1$ be an R-homomorphisem. Let $M_1 = K_2 \oplus K_3$, for some $K_3 \leq M_1$. Define $g: M_1 \to M_1$, by

$$g(x) = \begin{cases} g_1(x), & if\ x \in K_2 \\ 0\ , & if\ x \in K_3 \end{cases}$$

Clearly, g is an R-homomorphisem. Since M_1 is a PR-module, so $\ker g \leq_P M_1$.
Now $kerg = \{\alpha + \beta \in M_1;\ \ g(\alpha + \beta) = 0, \alpha \in K_2, \beta \in K_3\}$
$= \{\alpha + \beta \in M_1;\ \ \ g_1(\alpha) = 0,\ \alpha \in K_2, \beta \in K_3\} = \ker g_1 \oplus K_3$.
Hence $\ker g_1 \leq_P M_1$. But $\ker g_1 \subseteq K_2$, therefore, $\ker g_1 \leq_P K_2$. Thus K_2 is K_1-PR-module.
(2) ⇒ (3) Let K_1 and K_2 be a direct summand of M_1 and $g: M_1 \to K_1$ be an R-homomorphism. Since K_2 is K_1-PR-module and $g|_{K_2}: K_2 \to K_1$ be an R-homomorphism, then $\ker g|_{K_2} \leq_P K_2$.

(3) ⇒ (1) Clear (taking $K_1 = K_2 = M_1$, M_1 is M_1-PR-module and hence M_1 is a PR-module).

Proposition 3.6: Let M_1 be a pure simple R-module and let M_2 be an R-module. If M_1 is M_2-PR-module, then either
1- $Hom(M_1, M_2) = 0$ or,
2- Every nonzero R-homomorphism from M_1 to M_2 is a monomorphism.

Proof: Assume that $Hom(M_1, M_2) \neq 0$ and let $g: M_1 \to M_2$ be a non-zero R-homomorphism. Since M_1 is M_2-PR-module, then $\ker g \leq_P M_1$. But M_1 is pure simple, therefore $\ker g = \{0\}$. Thus g is a monomorphism.

Corollary 3.7: Let M_1 be a pure simple R-module and let M_2 be an R-module such that $Hom(M_1, M_2) \neq 0$. If M_1 is M_2-PR-module, then M_1 is a Quasi Dedekind module. In particular if M_1 is a PR-module, then M_1 is Quasi Dedekind.

Proof: By Proposition 3.6, there is a monomorphism $g: M_1 \to M_2$. Assume M_1 is not Quasi Dedekind R-module. Then there exists a nonzero homomorphism $g_1: M_1 \to M_1$ such that $\ker g_1 \neq 0$. Since g is a monomorphism, then $\ker(gog_1) = \ker g_1 \neq 0$. Since M_1 is M_2-PR-module, then $\ker(gog_1) \leq_P M_1$. But M_1 is pure simple, therefore, $\ker g_1 = M$. Thus $g_1 = 0$, which is a contradiction. Thus M_1 is a Quasi Dedekind.

Proposition 3.8: Let A, B, C be an R-modules. If A is C-PR-module and $g: A \to B$ be an epimorphism, then B is C-PR-module.

Proof: Let P be the class of pure submodules, see [16] and let $g_1: B \to C$ be an R-homomorphisem. Consider the short exact sequence $0 \to \ker g_1 \xrightarrow{i} B \xrightarrow{g_1} Im\, g_1 \to 0$.
To show that g_1 is P-epimorphism. Let $A \xrightarrow{g} B \xrightarrow{g_1} Im\, g_1$ be an R-homomorphisem and $Im\, g_1 \leq C$. Since A is C-PR-module, therefore, $\ker g_1 og \leq_P A$. Hence construct the short exact sequence $0 \to \ker g_1 og \xrightarrow{i} A \xrightarrow{g_1 og} Im\, g_1 og \to 0 \in P$.
Since $g_1 og$ is P-epimorphism, then g_1 is P-epimorphism.

Corollary 3.9: Let M_1 and M_2 be R-modules. If M_1 is M_2-PR-module, $\frac{M_1}{N}$ is M_2-PR-module, for every $N \leq M_1$.

Proof: Follows by Proposition 3.8, taking $M_1 = A$ and $\frac{M_1}{N} = B$.

Proposition 3.10: Let M_1 be an R-module, if R is M_1-PR-module, then every cyclic submodule of M_1 is flat. In particular if R is a PR-ring, then every principal ideal is flat, i.e. R is a principal flat ring PF-ring.

Proof: Let M_1 be an R-module, R be M_1-PR-module and let $m \in M_1$. Now consider the following short exact sequence $0 \to \ker f \xrightarrow{i} R \xrightarrow{g} Rm \to 0$,
where i is the inclusion map and g is a map define by $g(r) = rm, \forall r \in R$. Let $i_2: Rm \to M_1$, be the inclusion map. Since R is M_1-PR-module and $i_2 og: R \to M_1$, then $\ker(i_2 og) \leq_P R$. But i_2 is a monomorphism, therefore $\ker g = \ker(i_2 og)$. Thus $\ker g \leq_P R$. But R is a flat R-module, therefore Rm is flat by [8, Proposition 3.60].

Theorem 3.11: Let R be a ring. Then R is flat ring if and only if every flat R-module are relatively PR-module to any flat R-module.

Proof: Suppose that R is flat ring. Let M_1 and M_2 be a flat R-modules and let $g: M_1 \to M_2$ be R-homomorphism. Since $M_1 \oplus M_2$ is flat, then by [14] $C_{M_1} + T_g$ is flat submodule of $M_1 \oplus M_2$. Therefore M_1 is M_2-PR-module by Theorem 2.17.
For the opposite. Suppose that M_1be a flat R-module and $M_2 \leq M_1$. Then there exists a free R-module F and an epimorphisem $g: F \to M_2$, [11, Corollary 4.4.4, p.89]. Let $i: M_2 \to M_1$ be inclusion map. Consider the map $iog: F \to M_1$. Since F is flat, then by our assumption F is M_1-PR-module. Thus $\ker iog \leq_P F$ .But i is a monomorphism, so $\ker iog = \ker g \leq_P F$ .But F is flat, therefore M_2 is flat by [8, Proposition 3.60].

Theorem 3.12: Let R be a ring. The following statements are equivalent:
1- R is flat ring;
2- Every finitely generated flat R-module are relatively PR-module to any flat R-module;
3- Every finitely generated submodule of a finitely generated flat R-module is flat.

Proof: (1) $\Rightarrow$ (2) Follows by Theorem 3.11.
(2) $\Rightarrow$ (1) Let I be a finitely generated ideal in R. Then there exists a finitely generated free R-module F and an epimorphism $g: F \to I$, [11, Corollary 4.4.4, p.89]. Let $i: I \to R$ be the inclusion map. Consider the map $iog: F \to R$. Since F is a f.g flat R-module, then F is R-PR-module by (2) and hence $\ker iog \leq_P F$. Since i is monomorphism, then $\ker iog = \ker g \leq_P F$. But F is flat, therefore, I is flat by [8, Proposition 3.60, p.139].
(2) $\Rightarrow$ (3) Since R is flat ring, then (3) hold by [7].
(3) $\Rightarrow$ (2) Let M_1 be a finitely generated flat module and let $g: M_1 \to M_1$ be an R-homomorphism. Let K be a finitely generated submodule of $C_{M_1} + T_g$. Hence K is a finitely generated submodule of $M_1 \oplus M_1$. But M_1 is flat by (3), therefore $C_{M_1} + T_g$ is flat by [8, Proposition 3.60, p.139] and hence M_1 is PR-module by Theorem 2.17. Thus M_1 is M_1-PR-module.
(3) $\Rightarrow$ (1) Clear (3→2→1).

Theorem 3.13: Let R be a ring. The following statements are equivalent:
1- R is flat ring;
2- Every projective R-module are relatively PR-module to any flat R-module;
3- Every submodule of a projective R-module is flat.

Proof: (1) $\Rightarrow$ (2) Let M_1 be a projective R-module and hence flat and let M_2 be a flat R-module. Thus by Theorem 3.1 M_1 is M_2-PR-module.
(2) $\Rightarrow$ (1) Suppose that I be an ideal in R. There is a free R-module F and an epimorohism $g: F \to I$, [11, Corollary 4.4.4, p.89]. Let $i: I \to R$ be the inclusion map. Consider the map $iog: F \to R$. Since F is projective, then F is R-PR-module by (2). Hence $\ker g = \ker iog \leq_P F$. Therefore I is flat by [8, Proposition 3.60, p.139].
(2) $\Rightarrow$ (3) Assume that M_1 is a projective R-module and let $K \leq M_1$. There is a free R-module F and an epimorphism $g: F \to K$, [11, Corollary 4.4.4, p.89]. Let $i: K \to M_1$ be the inclusion map. Consider the map $iog: F \to M_1$. Since F is projective, then F is M_1-PR-module by (2). Therefore $\ker iog = \ker g \leq_P F$. Thus K is flat by [8, Proposition 3.60, p.139].
(3) $\Rightarrow$ (2) Suppose that M_1 is a projective R-module, then $M_1 \oplus M_1$ is projective and let $g: M_1 \to M_1$ be an R-homomorphism. Let $C_{M_1} + T_g \leq M_1 \oplus M_1$, then by (3) $C_{M_1} + T_g$ is flat. Therefore by Theorem 2.7, M_1 is PR-module and hence M_1 is M_1-PR-module.

(3) $\Rightarrow$ (1) Clear (3→2→1).

Theorem 3.14: R is flat ring if and only if $R\oplus R$ is R-PR-module.
Proof: Suppose that R is flat ring, then $R\oplus R$ is R-PR-module Theorem 3.11.
For the opposite, suppose that $R\oplus R$ is R-PR-module. Let $I = R_{a1} + R_{a2}$ be a two generated ideal in R. Define $g: R\oplus R \to I$ by $g(r_1, r_2) = r_1 a_1 + r_2 a_2, \forall r_1, r_2 \in R$. It is obvious that g is an epimorphism. Let $i: I \to R$ be the inclusion map. Consider the map $iog: R\oplus R \to R$. Since $R\oplus R$ is R-PR-module, then $\ker g = \ker iog \leq_P R\oplus R$. Therefore I is flat by [8, Proposition 3.60, p.139] and hence by [17] every finitely generated ideal in R is flat. Thus R is flat ring.

References

1. G. Ahmed, "Pure Rickart Modules and their Generalization," *International Journal of Mathematics Trends and Technology (IJMTT),* vol. 30, no.2, 2016.
2. G. Desale and W. K. Nicholson, "Endoprimitive rings," *J. Algebra*, vol. 70, pp. 548-560, 1981.
3. J. T. Knight, Commutative Algebra, Cambridge University Press, 1971.
4. A. S. Mijbass, "Quasi-Dedekind Modules," in *PhD. Thesis,* Colloge of Science, University of Baghdad, 1997.
5. D. J. Fieldhouse, "Pure simple and indecomposable rings," *Canadian Mathematical Bulletin,* vol. 13, no. 1, pp. 71-78, 1970.
6. Hattori, "A Fondation of Torsion Theory for Modules Over General Rings," *Nagoya Math. J.,* vol. 17, pp. 891-895, 1960.
7. F. H. Al-Alwan, "Modules on flat rings," M.Sc. Thesis, Collage of Science, Uinversity of Baghdad, 1985.
8. J. J. Rotman, An introduction to homological algebra, New York: Academic Press, 1979.
9. K. R. Goodearl, Ring Theory, Nonsingular Rings and Modules, New York: Marcel Dekker, 1979.
10. G. V. Wilson, "Modules with the summand intersection property," *Communications in Algebra,* vol. 14, pp. 21-38, 1986.
11. F. Kasch, Modules and Rings, London: Acad. Press., 1982.
12. M. Q. Rahman and B. H. Al-Bahrani, "On y-closed dual Rickart modules," *Iraqi Journal of science*, vol. 62(4), pp. 1226-1230, 2021.
13. M. Q. Rahman and B. H. Al-Bahrani, "On Rickart modules," *Iraqi Journal of science,* vol. 60(11), pp. 2473-2477.
14. S. M. Yassen, F-Regular Modules, M.Sc., Thesis: Colloge of Sience, University of Baghdad, 1993.
15. M. Q. Rahman and B. H. Al-Bahrani, "On y-closed Rickart modules," *Iraqi Journal of science,* vol. 61(10), pp. 2681-2686, 2020.
16. Y. Durgun and S. Ozdemir, "On S-closed submodules," *J. Korean Math. Soc.,* vol. 54, pp. 1281-1299, 2017.
17. A. G. Naoum, "On finitly generated projective and flat ideales on commutative rings," *Periodica mathematica,* pp. 251-260, 1985.