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On Weakly Second Submodules

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Abstract
Let M be a non-zero right module over a ring R with identity. The weakly second submodules is studied in this paper. A non-zero submodule N of M is weakly second Submodule when $Nab \subseteq K$, where a, $b \in R$ and K is a submodule of M implies either $Na \subseteq K$ or $Nb \subseteq K$. Some connections between these modules and other related modules are investigated and number of conclusions and characterizations are gained.

Keywords: weakly second submodules, S-weakly second submodules, weakly secondary submodules, second submodules, secondary submodules.

المقاسات الجزئية الثنائية الضعيفة

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الخلاصة
ليكن M مقاسا ايمنا على حلقة R ذات محايد. في هذا البحث درسنا المقاسات الثنائية الضعيفة. نطلق على مقاس جزئي $N \neq 0$ من M بانه ثنائي ضعيف عندما لكل $a, b \in R$ و K مقاس جزئي من M بحيث ان $abN \subseteq K$ فانه اما $aN \subseteq K$ او $bN \subseteq K$. عدد من النتائج والمكافئات لهذا المفهوم اعطيت وكذللك دراسة العلاقات بين هذه المقاسات والمقاسات الاخرى.

1. Introduction

R is denoted a ring has an identity and M is studied as a non-zero left S-right R-bimodule where $S = \text{End}_R(M)$ the endomorphism ring of M. We use the notation " $\subseteq$ " to denote inclusion. $0 \neq N$ is said to be a second submodule of M if for any $a \in R$, the endomorphism $f_a: N \to N$ defined by $f_a(n) = na$ for each $n \in N$, is either surjective or zero (that is $Imf_a = Na = N$ or $Imf_a = Na = 0$) [1]. Equivalently $0 \neq N$ is a second submodule of M if $NI = N$ or $NI = 0$ for every ideal I of R [1]. In that situation, $ann_R(N)$ is a prime ideal of R[1]. A non-zero module M is a second (or coprime) if M is a second submodule of itself [1]. As a new type of second submodules, the concept of weakly second submodules was presented and studied in [2]. A non-zero submodule N of M is weakly second submodule whenever $Nab \subseteq K$ where a, $b \in R$ and K a submodule of M implies either $Na \subseteq K$ or $Nb \subseteq K$ [2]. A non-zero module M is a weakly second module if M is a weakly second submodule of itself [2]. In fact this idea as a dual notion of the concept weakly prime (sometimes is called classical prime) submodules. A proper submodule N of M is wekly prime whenever $Kab \subseteq N$ where a, $b \in R$ and K a submodule of M implies either $Ka \subseteq N$ or $Kb \subseteq N$ [3]. In [4], we provide the idea of weakly secondary as a generalization of weakly second concept and in the same time it is a new type of secondary submodules and a dual notion of classical primary submodules respectively. A nonzero

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submodule N of M is weakly secondary submodule whenever $Nab \subseteq K$ where $a, b \in R$ and K is a submodule of M implies either $Na \subseteq K$ or $Nb^t \subseteq K$ for some positive integer t [4]. $0 \neq N$ is a secondary submodule of M if for any $a \in R$, the endomorphism $f_a: N \to N$ defined by $f_a(n) = na$ for each $n \in N$, is either surjective or nilpotent (that is $Imf_a = Na = N$ or $Imf_a = Na^t = 0$ for some positive integer t) [1]. Equivalently, $0 \neq N$ is secondary of M if for every ideal I of R, $NI = N$ or $NI^t = 0$ for some positive integer t [1]. A proper submodule K of M is classical primary whenever $Nab \subseteq K$ where $a, b \in R$ and N is a submodule of M then $Na \subseteq K$ or $Nb^t \subseteq K$ for some positive integer t [5]. N is called simple (sometimes minimal) submodule of a module M if $N \neq 0$ and for each submodule L of M and N contains L properly implies $L = 0$ [6]. A module M is called simple module if M is simple submodule of itself [6]. M is coquasi-dedekind if all non-zero endomorphism of M is epimorphism (in other word, $f(M) = M$ for every $0 \neq f \in S$) [7]. Let R be a commutative integral domains, M is called divisible module over R if $Ma = M$ for each $0 \neq a \in R$ [6]. A proper submodule N is maximal if it is not properly contained in any proper submodule of M [6]. A proper submodule N is called prime if $mr \in N$ implies $m \in N$ or $Mr \subseteq N$ [8]. A proper ideal I is prime if $ab \in I$ where $a, b \in R$ implies $a \in I$ or $b \in I$ [9]. Equivalently, a proper ideal I is prime if $AB \subseteq I$ where A and B are ideals of R implies $A \subseteq I$ or $B \subseteq I$ [9]. A ring in which every ideal is prime is called fully prime[10]. Equivalently, a ring R is fully prime if and only if it is fully idempotent and the set of ideals of R is totally ordered under inclusion [10]. M is comultiplication provided that for each submodule N of M, there exists an ideal I of R such that $[0:_M I] = ann_M(I) = \{m \in M \text{ and } Im = 0\}$ is a submodule of M [11]. We able to take $I = [0:_R N] = ann_R(N) = \{r \in R \text{ and } Nr = 0\}$ is an ideal of R [11]. N is called a submodule pure in an R-module M when $NI = MI \cap N$ for each ideal I of R[12]. M is called regular when every submodule of M is pure [12]. M is called S-second if every $f \in S$ implies $f(M) = M$ or $f(M) = 0$ [13]. M is indecomposable if $M \neq 0$ and it cannot be written as a direct sum of non-zero submodules (that is 0 and M are the only direct summands) [6]. M is called multiplication when each submodule N of M, we have $N = MI$ for an ideal I of R [14]. We able to take $I = [N :_R M] = \{r \in R \text{ and } Mr \subseteq N\}$ [14]. M is a scalar module when for each $f \in End(M)$ there is $a \in R$ with $f(m) = ma$ for all $m \in M$ [15]. Other studies within [16-26] is related topics.

The paper consists of five parts. Within part two, we investigate the weakly second submodules idea and we supply examples (Remarks and Examples 2.3) and needful features of this concept. We add a new characterization (Proposition 2.9) and some properties of this concept (Proposition 2.4). The direct sum of weakly second submodules is discussed (Proposition 2.5). In Section three more characterizations is given (Theorem 3.1, Theorem 3.7 and Theorem 3.8). In section four we look for any relationships between weakly second submodules and related modules such as (Proposition 4.1 and Proposition 4.4). S-weakly second modules is dfined and basic properties about this modules is studied in section five. In what follows, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_{p^\infty}$, $\mathbb{Z}_n = \frac{\mathbb{Z}}{n\mathbb{Z}}$ and $Mat_n(R)$ we denote respectively, integers, rational numbers, the p-Prüfer group, the residue ring modulo n and an $n \times n$ matrix ring over R .

2. Weakly Second Submodules

Main facts of this part are introduced. We begin by the following.

Definition 2.1 [2] A nonzero submodule N of M is a weakly second submodule whenever $Nab \subseteq K$, where $a, b \in R$ and K is a submodule of M implies either $Na \subseteq K$ or $Nb \subseteq K$.

Theorem 2.2 [2] The following statements are equivalent

(1) N is a weakly second submodule of M.

(2) $N \neq 0$ and for each $a, b \in R$ implies $Nab = Na$ or $Nab = bN$.

Remarks and Examples 2.3

(1) Every second submodule is weakly second.

Proof

Let N be second of M then $N \neq 0$. Let $a, b \in R$ and K a submodule of M with $Nab \subseteq K$. By hypothesis $Nab = N$ or $Nab = 0$. In case $Nab = N$ implies $Na \subseteq N = Nab \subseteq K$. In case $Nab = 0$ implies $Na = Nab = 0 \subseteq K$ or $Nb = Nab \subseteq K$ as desired.

(2) Weakly second submodules fail to be second. Consider $M = \mathbb{Z}_P \oplus \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module where p is a prime number then M is weakly second since $Mab = Ma$ or $Mab = Ma$ for each $a, b \in \mathbb{Z}$ but M is not second since if $a = p$ then $pM = 0 \oplus \mathbb{Z}_{p^\infty}$

(3) As another example of (2), the submodule $N =< \frac{1}{p} + \mathbb{Z} >\oplus \mathbb{Z}_{p^\infty}$ is weakly second of $M = \mathbb{Z}_{P^\infty} \oplus \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module but N is not second.

(4) Clearly every weakly second submodule is weakly secondary while the converse is not true by [3].

(5) Clearly weakly second and weakly secondary concepts are coincide over Boolean rings.

(6) The secondary submodules and weakly second concepts do not imply from each one to another. The $\mathbb{Z}$-module $\mathbb{Z}_4$ is secondary since $\mathbb{Z}_4.a = \mathbb{Z}_4$ or $\mathbb{Z}_4.a^n = 0$ for some n a positive integer but $\mathbb{Z}_4$ is not weakly second because $\mathbb{Z}_4.2.2 = 0$ while $\mathbb{Z}_4.2 \neq 0$. On the other side, $M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module is weakly second but not secondary. Since for each $a, b \in \mathbb{Z}$, if a and b are not multiple of p implies $M.a.b = M \Rightarrow M.a^n = M$ for each positive integer n but when a or b is a multiple of p, we have $M.a.b = 0 \oplus \mathbb{Z}_{p^\infty} = K \Rightarrow Ma = K$ and $0_M \neq M.a^n \neq M$ for each positive integer n.

(7) The following implication is clear simple submodule $\Rightarrow$ second submodule $\Rightarrow$ weakly second.

(8) The following implication is clear coquasi-dedekind module $\Rightarrow$ divisible module $\Rightarrow$ second module $\Rightarrow$ weakly second module.

(9) It is clear $\mathbb{Z}_{p^\infty}$ and $\mathbb{Q}$ as $\mathbb{Z}$-modules are coquasi-dedekind (and hence are divisible) by (8) they are weakly second. Further it is well known that every direct summand of divisible module is divisible [6]. And every product (or sum) of divisible modules is divisible[6]. Accordingly, $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$ (where p and q prime numbers) and $\mathbb{Q} \oplus \mathbb{Q}$ as $\mathbb{Z}$-modules are divisible and hence weakly second.

(10) If M is weakly second module then M need not be coquasi-dedekind. For example $\frac{\mathbb{Q}}{\mathbb{Z}} \cong \oplus \sum_p \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module is divisible and hence it is weakly second but it is not coquasi-dedekind.

(11) If N is a maximal (and hence prime) submodule then N may not be weakly secondary. For example, $N = \mathbb{Z}_{12}.2$ is a maximal submodule in $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module but N is not weakly second since $N.2.3 = 0$ and neither $N.2 \neq 0$ nor $N.3 \neq 0$.

(12) Let N and H be submodules of an R-module M with $N \subseteq H \subseteq M$. If N is weakly second then H need not be weakly second. For example, let $N = \{\bar{0}, \bar{2}, \bar{4}\}$ and $H = \mathbb{Z}_6 = M$ submodules of $M = \mathbb{Z}_6$ as $\mathbb{Z}$-module where N is a simple submodule so it is weakly second while H is not weakly second because $H.2.3 = 0$ and $H.2 = N$ and $H.3 = \{\bar{0}, \bar{3}\}$.

(13) Let N and H be submodules of an R-module M with $N \subseteq H \subseteq M$. If H is weakly second then N need not be weakly second submodule of M. For example, let $N =< \frac{1}{p} + \mathbb{Z} >\oplus< \frac{1}{q} + \mathbb{Z} >$ and $H = M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$ be submodules of $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$ as $\mathbb{Z}$-module where p and q prime numbers. Since M is a divisible module then M is weakly second but N is not weakly second because $Np.q = 0_M$ while $N.p = 0 \oplus \mathbb{Z}_{q^\infty}$ and $N.q = \mathbb{Z}_{p^\infty} \oplus 0$.

(14) As another example of (13), $\mathbb{Q}$ as $\mathbb{Z}$-module is divisible so it is weakly second but the submodule $\mathbb{Z}$ is not weakly second.

Proposition 2.4 Every nonzero homomorphic image of weakly second submodule is weakly second.

Proof

Let A and B be R-modules and $0 \neq \nabla: A \to B$ an R-homomorphism. Let N be a weakly second of A. Firstly since $\nabla \neq 0$ implies $\nabla(N) \neq 0$. For each $a, b \in R$ then $\nabla(N)ab = \nabla(Nab) = \nabla(Na) = \nabla(N)a$ or $\nabla(N)ab = \nabla(Nab) = (Nb) = \nabla(N)b$.

Proposition 2.5 Let $\mathbb{A}$ and $\mathbb{B}$ be non-zero submodules of of R-modules M_1 and M_2 respectively. If $N = \mathbb{A} \oplus \mathbb{B}$ is a weakly second of $M = M_1 \oplus M_2$ then $\mathbb{A}$ and $\mathbb{B}$ are weakly second submodules of R-modules M_1 and M_2 respectively

Proof

First $\mathbb{A} \neq 0_{M_1}$ and $\mathbb{B} \neq 0_{M_2}$ because $N \neq 0_M$. Let $a, b \in R$ then either $(\mathbb{A} \oplus \mathbb{B})ab = (\mathbb{A} \oplus \mathbb{B})a$ or $(\mathbb{A} \oplus \mathbb{B})ab = (\mathbb{A} \oplus \mathbb{B})b$ and hence $\mathbb{A}ab = \mathbb{A}a$ or $\mathbb{B}ab = \mathbb{B}b$ and $\mathbb{A}ab = \mathbb{A}b$ or $\mathbb{B}ab = \mathbb{B}b$ as required. ■

Corollary 2.6 Every non-zero summand of a weakly second module is weakly second.

Remarks and Examples 2.7

(1) The direct sum of weakly second submodules need not be weakly second. For example, $\mathbb{Z}_p$ and $\mathbb{Z}_q$ as $\mathbb{Z}$-modules are weakly second where p and q are prime numbers then $\mathbb{Z}_p \oplus \mathbb{Z}_q$ is not weakly second $\mathbb{Z}$-module since $(\mathbb{Z}_p \oplus \mathbb{Z}_q)pq = 0 \oplus 0$ while $(\mathbb{Z}_p \oplus \mathbb{Z}_q)p = 0 \oplus \mathbb{Z}_q$ and $(\mathbb{Z}_p \oplus \mathbb{Z}_q)q = \mathbb{Z}_p \oplus 0$.
(2) In general $\mathbb{Z}_n \oplus \mathbb{Z}_m$ as $\mathbb{Z}$-module is not weakly second for each positive integers $n \neq m$.
(3) Obviously, if n is a square-free integer (an integer which has a prime factorization has exactly one factor for each prime that appears in it) then $\mathbb{Z}_n$ as $\mathbb{Z}$-module is not weakly second. Oppositely fails in general, $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module is not weakly second because $\mathbb{Z}_{12}.3.4 = 0$ but $\mathbb{Z}_{12}.3 \neq 0 \neq \mathbb{Z}_{12}.4$ and 12 is not square-free.
(4) Let $M = A \oplus B$ be a direct sum of two R-modules A and B. If N is a weakly second submodule of A then $N \oplus B$ may be not a weakly second submodule of M. For example $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module so it is weakly second while $\mathbb{Q} \oplus \mathbb{Z}$ is not a weakly second since $[\mathbb{Q} \oplus 6\mathbb{Z} :_{\mathbb{Z}} \mathbb{Q} \oplus \mathbb{Z}] = 6\mathbb{Z}$ is not a prime ideal of $\mathbb{Z}$ then by Theorem, $\mathbb{Q} \oplus \mathbb{Z}$ is not a weakly second $\mathbb{Z}$-module. In fact for any R-module M then $M \oplus \mathbb{Z}$ is not a weakly second $\mathbb{Z}$-module.
(5) Let $M = A \oplus B$ be a direct sum of two R-modules A and B. If N is divisible (or weakly second) of A and H is not weakly second of B then $N \oplus H$ is not weakly second of M.

Proof

Suppose $N \oplus H$ is a weakly second submodule of M then for each $a, b \in R$ we have $(N \oplus H)ab = (N \oplus H)a$ or $(N \oplus H)ab = (N \oplus H)b$. It follows $Hab = Ha$ or $Hab = Hb$ which is a contradiction because H is not a weakly second submodule of B as desired.

(6) $\mathbb{Q} \oplus \mathbb{Z}$, $\mathbb{Q} \oplus \mathbb{Z}_n$, $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ and $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_n$ as $\mathbb{Z}$-modules are not weakly second by (4) where n is a square-free integer.

Proposition 2.8 If N is a weakly second submodule of M then $N \oplus N$ is a weakly second submodule of $M \oplus M$ as R-module.

Proof

Firstly $N \oplus N \neq 0 \oplus 0$ because $N \neq 0$. Let $a, b \in R$ then $(N \oplus N)ab = Nab \oplus Nab$ but N is weakly second implies either $Nab = Na$ or $Nab = Nb$ and hence $(N \oplus N)ab = (N \oplus N)a$ or $(N \oplus N)ab = (N \oplus N)b$ as required.

Proposition 2.9 The next are equivalent
(1) N is a weakly second submodule of M.
(2) $\frac{N}{H}$ is a weakly second submodule of $\frac{M}{H}$ for each submodule H of M contained in N.

Proof

(1) $\Rightarrow$ (2) Let N be a weakly second submodule M and $\pi: M \to \frac{M}{H}$ be the natural homomorphism for each submodule H of M contained in N so by Proposition, $\pi(N) = \frac{N}{H}$ is a weakly second submodule $\frac{M}{H}$.
(2) $\Rightarrow$ (1) It is clear by taking $H = 0$.

3. More Characterizations and Facts About Weakly Second Idea.

Theorem 3.1 The next statements are equivalent
(1) N is a weakly second submodule of an R-module M.
(2) $N \neq 0$ and $[K:_R N]$ is a prime ideal of R for each submodule $K \not\supseteq N$ in M.

Proof

(1) $\Rightarrow$ (2) Assume N is a weakly second and K a submodule of M with $N \nsubseteq K$ implies $[K:_R N] \neq R$. Let $a, b \in R$ with $ab \in [K: N]$ implies $Nab \subseteq K$ then $Na \subseteq K$ or $Nb \subseteq K$ so either $a \in [K:_R N]$ or $b \in [K:_R N]$ as required.
(2) $\Rightarrow$ (1) Let $Nab \subseteq K$ where $a, b \in R$. In case $N \subseteq K$ then already $Na \subseteq K$ and $Nb \subseteq K$. If $N \nsubseteq K$ then $[K:_R N]$ is prime of R by hypothesis and $ab \in [K:_R N]$ implies $Na \subseteq K$ or $Nb \subseteq K$ as desired.

Corollary 3.2 Every submodule of a module over a fully prime ring is weakly second.

Proof

Directly by Theorem 3.1 of (2) $\Rightarrow$ (1)

Corollary 3.3 If N is a weakly second submodule of M then $ann_R(N)$ is prime of R.

Proof

Directly via Theorem 3.1 of (1) $\Rightarrow$ (2)

Examples 3.4 The opposite of Corollary 3.3 is not hold in general since $ann_R(N) = 0$ a prime ideal of $\mathbb{Z}$ for every non-zero submodule N of $\mathbb{Z}$ while N is not weakly second.

Corollary 3.5 If N is a weakly second submodule of an M then for every submodule $K \not\supseteq N$ in M we have $[K:_R N] = [K:_R Nb]$ for each $b \in R$ with $b \notin [K:_R N]$.

Proof

Let $a \in [K:_R N]$ then $Na \subseteq K$ implies for each $b \in R$ $Nab \subseteq K$ so $a \in [K:_R Nb]$. Conversely, let $a \in [K:_R Nb]$ then $Nab \subseteq K$ and so $ab \in [K:N]$. Via Theorem 3.1, $[K:N]$ is prime of R and $b \notin [K:_R N]$, implies that $a \in [K:_R N]$ as required.

Corollary 3.6 If N is a weakly second of M then $ann_R(N) = ann_R(bN)$ for each $b \in R$ with $b \notin ann_R(N)$.

Proof

Directly by Corollary 3.5

Theorem 3.7 The following statements are equivalent

(1) is a weakly second of M.

(2) The set $\{[\mathcal{Q}:_R \mu], \mathcal{Q}$ is a submodule of M with $\mathcal{Q} \not\supseteq \mu\}$ is a chain of prime ideals of R.

Proof

(1) $\Rightarrow$ (2) Initially $[\mathcal{Q}:_R \mu]$ is prime of R for each submodule $\mathcal{Q} \not\supseteq \mu$ in M by Theorem 3.1. Let $\mathcal{Q}$ and ϱ be submodules of M, $\mathcal{Q} \not\supseteq \mu$ and $\varrho \not\supseteq \mu$ then $[\mathcal{Q}:_R \mu]$ and $[\varrho:_R \mu]$ are prime ideals of R. Suppose $[\mathcal{Q}:_R N] \nsubseteq [\varrho:_R \mu]$ and $[\varrho:_R \mu] \nsubseteq [\mathcal{Q}:_R \mu]$ this means there exist ideals I and J of R with $I \subseteq [\mathcal{Q}:_R N]$, $I \nsubseteq [\varrho:_R \mu]$, $J \subseteq [\varrho:_R \mu]$ and $J \nsubseteq [\mathcal{Q}:_R \mu]$. So $IJN \subseteq \mathcal{Q}$ and $IJN \subseteq \varrho$ implies $IJ \subseteq [\mathcal{Q} \cap \varrho:_R \mu]$. Since $\mathcal{Q} \cap \varrho \not\supseteq \mu$ then $[\mathcal{Q} \cap \varrho:_R \mu]$ is prime of R it follows $I \subseteq [\mathcal{Q} \cap \varrho:_R \mu]$ or $J \subseteq [\mathcal{Q} \cap \varrho:_R \mu]$. If $I \subseteq [\mathcal{Q} \cap \varrho:_R \mu]$ we have $I \subseteq [\mathcal{Q}:_R \mu]$ and $I \subseteq [\varrho:_R \mu]$. If $J \subseteq [\mathcal{Q} \cap \varrho:_R \mu]$ then $J \subseteq [\mathcal{Q}:_R \mu]$ and $J \subseteq [\varrho:_R \mu]$. So we see in any case we have a contradiction.

(2) $\Rightarrow$ (1) By Theorem 3.1.

Theorem 3.8 The next are equivalent

(1) N is a weakly second submodule of an R-module M.

(2) $N \neq 0$ and for each ideals I, J of R and K a submodule of M such that $IJN \subseteq K$ implies $IN \subseteq K$ or $JN \subseteq K$.

Proof

(1) $\Rightarrow$ (2) First N is a weakly second of an R-module M then $N \neq 0$. Let I and J be ideals of R and K a submodule of M. If $N \nsubseteq K$ we have either $IJN \nsubseteq K$ and so nothing to prove or $IJN \subseteq K$ it follows $IJ \subseteq [K:_R N]$ and by Theorem ,$[K:_R N]$ is a prime ideal so $I \subseteq [K:_R N]$ or $J \subseteq [K:_R N]$ and hence $IN \subseteq K$ or $JN \subseteq K$. In case $N \subseteq K$ then the result already is obtained.

(2) $\Rightarrow$ (1) Let $abN \subseteq K$, where $a, b \in R$ and K a submodule of M, then $< a >< b > N \subseteq K$ By hypothesis either $< a > N \subseteq K$ or $< b > N \subseteq K$ that is $aN \subseteq K$ or $bN \subseteq K$ as desired.

Corollary 3.9 The following statements are equialent

(1) N is a weakly second submodule of an R-module M.

(2) $N \neq 0$ and for each ideals I and J of R implies $IN = IJN$ or $JN = IJN$.

Proof

Similarly to the proof of Theorem 2.2 and by Theorem 3.1.

Corollary 3.10 The following statements are equialent

(1) N is a weakly second of an R-module M.

(2) $N \neq 0$ and for each ideals I and J of R and K a submodule of M such that $N \nsubseteq K$ and $IJ \subseteq [K:N]$ implies $I \subseteq [K:N]$ or $J \subseteq [K:N]$.

Proof.

Directly via corollary 3.9 and Theorem 3.1

Corollary 3.11 The following statements are equialent

(1) N is a weakly second of an R-module M.

(2) $N \neq 0$ and for each ideals I and J of R and K a submodule of M with $N \nsubseteq K, IJ \subseteq [K:_R N]$ and $I \nsubseteq [K:_R N]$ implies that $J \subseteq [K:_R N]$.

Proof

Directly via Theorem 3.8

Corollary 3.12 The following statements are equialent

(1) N is a weakly second submodule of an R-module M.

(2) $N \neq 0$ and for each ideals I and J of R and K a submodule of M such that $N \nsubseteq K$, $IJ \subseteq [K:_R N]$ and $[K:_R N] \subset I$ implies that $J \subseteq [K:_R N]$.

Proof

Directly via Theorem 3.8

4. Weakly Second Submodules and Related Concepts

The following result is given in [11], we give the details of the proof.

Proposition 4.1 If N is a non-zero comuliplication submodule of M together with $ann_R(N)$ is prime of R then N is second.

Proof

Let $N \neq 0$. For every $a \in R$ we can define the endomorphism $f_a : N \to N$ by $f_a(n) = na$ for each $n \in N$ then $Imf_a = Na$. Because N is comultiplication implies $Na = ann_N(I)$ for an ideal I of R so $NaI = 0$ follows $aI \subseteq ann_R(N)$. But $ann_R(N)$ is prime so $Na = 0$ or $NI = 0$. In case $Na \neq 0$ then $NI = 0$ follows $Na = ann_N(I) = N$ as desired.

Corollary 4.2 Let M be a comuliplication R-module such that the annihilator of any non-zero submodule of M is a prime ideal of R then every nonzero submodule is second.

Proof

Because every submodule of a comultiplication module is comultiplication then by Proposition 4.1, the result is obtained.

Corollary 4.3 Let N be a non-zero comuliplication submodule of M. Discuss the equivalent below

(1) N is a weakly second submodule of M.

(2) $ann_R(N)$ is a prime ideal of R.

(3) N is a second submodule.

Proof

(1) $\Rightarrow$ (2) From Corollary 4.2, (2) $\Rightarrow$ (3) Via [1] and (3) $\Rightarrow$ (1) is clear.

Proposition 4.4 Every non-zero pure submodule of a weakly second module is weakly second.

Proof

Let N be a non-zero pure submodule of a weakly second R-module M. Then for each ideals I and J of R implies $MIJ = MI$ or $MIJ = MJ$. It follows either $NIJ = N \cap MIJ = N \cap MI = NI$ or $NIJ = N \cap MIJ = N \cap MJ = NJ$ as desired.

Corollary 4.5 Each submodule of a regular weakly second module is weakly second.

Corollary 4.6 Any submodule of a semisimple weakly second module is weakly second.

Example 4.7 $\mathbb{Z}_6$ as $\mathbb{Z}$-module is semisimple but not weakly second as shown in Remark and Example 2.3 (12) confirms that the status weakly second in Corollary 4.6 can not omitted.

5. *S*-Weakly Second Modules

At this point we define S-weakly second modules. Firstly we supply a characterization and examples of S-second modules.

Theorem 5.1 The following are equivalent

(1) M is an S-second module.

(2) $M \neq 0$ and whenever $\zeta(M) \subseteq K$ where $\zeta \in S$ and K a submodule of M implies either $M = K$ or $\zeta(M) = 0$.

Proof

(1) $\Rightarrow$ (2) Assume M is an S-second R-module then $M \neq 0$. Let $\zeta(M) \subseteq K$ for some $\in S$ and K a submodule of M. By hypothesis either $\zeta(M) = M$ or $\zeta(M) = 0$ implies $M = K$ or $\zeta(M) = 0$.

(2) $\Rightarrow$ (1) By (2) we can choose $K = \zeta(M)$ where $\zeta \in S$ implies $\zeta(M) \subseteq \zeta(M)$ and hence $\zeta(M) = M$ or $\zeta(M) = 0$.

Remarks and Examples 5.2

(1) Every S-second module is second.

Proof

Let M be S-second then for every $f \in S$, either $f(M) = M$ or $f(M) = 0$. For each $a \in R$, define $f_a: M \to M$ by $f_a(m) = ma$ for every $m \in M$ and it is well known $f_a \in S$ and $Imf_a = Ma$. By hypothesis $Ma = M$ or $Ma = 0$ as desired.

(2) The opposite of (2) is not valid in general. $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module is second but not S-second because there is an endomorphism

$f = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \in S = End_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \begin{pmatrix} End_{\mathbb{Z}}(\mathbb{Z}_2) & Hom_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) \\ Hom_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) & End_{\mathbb{Z}}(\mathbb{Z}_2) \end{pmatrix} \cong Mat_2(\mathbb{Z}_2) = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$ and $f(x, y) = (x, 0)$ for each $(x, y) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ implies $\bar{0} \oplus \bar{0} \neq Imf = \mathbb{Z}_2 \oplus \bar{0} \neq \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(3) Every S-Second module is indecomposable (that is when a module M has a decomposition then M is not S-second).

Proof

Let M be an S-second R-module then $M \neq 0$. Suppose that $M = A \oplus B$ for some R-modules A and B. So we can define the map $\zeta: M \to M$ maps $\zeta: M \to M$ by $\zeta(x, y) = (x, 0)$ then $\zeta \in S$ implies $0 \neq \zeta(M) = A \oplus 0 \neq M$ and hence M is not S-second which is a contradiction.

(4) The counter of (3) is not correct comprehensively. $\mathbb{Z}$ and $\mathbb{Z}_4$ as $\mathbb{Z}$-modules are indecomposable but not second and hence it is not S-second.

(5) Evidently coquasi-dedekind module is S-second.

(6) $\frac{\mathbb{Q}}{\mathbb{Z}} \cong \oplus \sum_p \mathbb{Z}_{p^\infty}$ is not S-second since if not then $\frac{\mathbb{Q}}{\mathbb{Z}}$ is indecomposable via (3) which is a contradiction and hence $\frac{\mathbb{Q}}{\mathbb{Z}}$ is not coquasi-dedekind.

(7) Obviously every simple module is S-second.

Definition 5.3 A non-zero R-module M is called S-weakly second whenever $\zeta\vartheta(M) \subseteq K$, where ζ, $\vartheta \in S$ and K a submodule of M implies either $\zeta(M) \subseteq K$ or $\vartheta(M) \subseteq K$.

Remarks and Examples 5.4

(1) Every S-weakly second module is weakly second.

Proof

Let M be an S-weakly second R-module then $M \neq 0$. Let $Mab \subseteq K$ for some a, $b \in R$ and K a submodule of M. Define the endomorphisms $f_a: M \to M$ by $f_a(m) = ma$ and $g_b: M \to M$ by $g_b(m) = mb$ for each $m \in M$. Then $fg(M) = f\big(g(M)\big) = f(Mb) = f(M)b = Mab \subseteq K$. By hypothesis either $f(M) \subseteq K$ or $g(M) \subseteq K$ that is $Ma \subseteq K$ or $Mb \subseteq K$ as desired.

(2) Reversely of (1) fails in general, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module is second (and hence weakly second) but it is not S-weakly second since if we take $f = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}$ and $g = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix} \in S = End_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \cong Mat_2(\mathbb{Z}_2)$ implies $fg(M) = \{fg\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \bar{0} \\ \bar{0} \end{pmatrix}$ for each $(\bar{x}, \bar{y}) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \} = \bar{0} \oplus \bar{0}$ while $f(M) = \mathbb{Z}_2 \oplus \bar{0}$ and $g(M) = \bar{0} \oplus \mathbb{Z}_2$.

(3) Every S-weakly second module is indecomposable (that is when a module M has a decomposition then M is not S-weakly second).

Proof

Let M be an S-weakly second R-module then $M \neq 0$. Suppose that $M = A \oplus B$ for some R-modules A and B. So we can define the maps $\zeta: M \to M$ $\zeta(x, y) = (x, 0)$ and $\vartheta: M \to M$ by $\zeta(x, y) = (0, y)$ for each $(x, y) \in M$. It is clear that ζ, $\vartheta \in S$ implies $\zeta\vartheta(M) = \zeta(\vartheta(M)) = \zeta(0 \oplus B) = 0 \oplus 0$ but $\zeta(M) = A \oplus 0$ and $\zeta(M) = 0 \oplus B$. Hence M is not S-weakly second which is a contradiction.

(4) The inverse of (3) is not hold in general, $\mathbb{Z}$ and $\mathbb{Z}_4$ as $\mathbb{Z}$-module are indecomposable but not S-weakly second.

(5) Every S-second module is S-weakly second.

Proof

Let M be an S-second R-module then $M \neq 0$. Let ζ, $\vartheta \in S$ and K a submodule of M with $\zeta\vartheta(M) \subseteq K$. By hypothesis $\zeta\vartheta(M) = M$ or $\zeta\vartheta(M) = 0$. In case $\zeta\vartheta(M) = M$ implies $\zeta(M) \subseteq M = \zeta\vartheta(M) \subseteq K$. In case $\vartheta\zeta(M) = 0$ implies $\zeta(M) = \zeta\vartheta(M) = 0 \subseteq K$ or $\zeta(M) = 0 \subseteq K$ as desired.

(6) Oppositely of (5) is not correct generally. Let F be a field and let R be the set of infinite matrices over F that have the form $\begin{pmatrix} A & & 0 & \\ & a & & \\ 0 & & a & \\ & & & \ddots \end{pmatrix}$ Where A is any finite matrix and a is any element of F. It is not hard to see that R is a ring with identity and the only non-zero proper ideal I of R is the subset of all matrices of R of the form $\begin{pmatrix} A & & 0 & \\ & 0 & & \\ 0 & & 0 & \\ & & & \ddots \end{pmatrix}$ so is clear $I = I^2$ and hence I is prime [10], also it is obvious the zero ideal is prime and hence $R \cong End(R)$ is fully prime ring. Via Theorem 3.1, R is a weakly second which is not second.

(7) We have the implication coquasi-dedekind modules $\Rightarrow$ S-second modules $\Rightarrow$ S-weakly second modules $\Rightarrow$ indecomposable modules.

Theorem 5.5 Study the equivalent

(1) M is an S-weakly second R-module.

(2) $M \neq 0$ and for each $\zeta, \vartheta \in S$ implies $\zeta\vartheta(M) = \zeta(M)$ or $\zeta\vartheta(M) \supseteq \vartheta(M)$.

Proof

(1) $\Rightarrow$ (2) Assume M is an S-weakly second R-module then $M \neq 0$. Let , $\vartheta \in S$ and $\zeta\vartheta(M) \subseteq K$ for submodule K of M. We can choose $K = \zeta\vartheta(M)$ so by (1) $\zeta(M) \subseteq K$ or $\vartheta(M) \subseteq K$ and hence $\zeta\vartheta(M) = \zeta(M)$ or $\zeta\vartheta(M) \supseteq \vartheta(M)$.

(2) $\Rightarrow$ (1) Let $M \neq 0$ and $\zeta, \vartheta \in S$ with $\zeta(M) \subseteq K$ for submodule K of M. By (2), $\zeta(M) = \zeta\vartheta(M) \subseteq K$ or $\vartheta(M) \subseteq \zeta\vartheta(M) \subseteq K$ as desired.

Corollary 5.6 If S is commutative ring we have the equivalent

(1) M is an S-weakly second R-module.

(2) $M \neq 0$ and for each $\zeta, \vartheta \in S$ implies $\zeta\vartheta(M) = \zeta(M)$ or $\zeta\vartheta(M) = \vartheta(M)$.

Proof

It is obvious

Theorem 5.7 The following statements are equivalent

(1) M is an S-weakly second R-module..

(2) $M \neq 0$ and $[K:_S M]$ is a prime ideal of S for each proper submodule K of M.

Proof

(1) $\Rightarrow$ (2) Assume M is S-weakly second and K a proper submodule of M implies $[K:_S M] \neq R$. Let ζ, $\vartheta \in S$ with $\zeta\vartheta \in [K:_S M]$ implies $\zeta\vartheta(M) \subseteq K$ then $\zeta(M) \subseteq K$ or $\vartheta(M) \subseteq K$ so either $\zeta \in [K:_S M]$ or $\vartheta \in [K:_S M]$ as required.

(2) $\Rightarrow$ (1) Let K be submodule of an R-module M such that $\zeta\vartheta(M) \subseteq K$ where $\zeta, \vartheta \in S$. In case $M = K$ then already $\zeta(M) \subseteq K$ and $\vartheta(M) \subseteq K$. If $M \neq K$ then $[K:_S M]$ is prime of S by hypothesis and $\zeta\vartheta \in [K:_S M]$ implies $\zeta(M) \subseteq K$ or $\vartheta(M) \subseteq K$ as desired.

Corollary 5.8 If M is an S-weakly second R-module M then $ann_S(M) = \{f \in S: f(M) = 0\}$ is prime of S.

Examples 5.9

(1) The opposite of corollary 5.8 is not hold in general. $ann_S(\mathbb{Z}) = 0$ is a prime ideal of $S = End_{\mathbb{Z}}(\mathbb{Z}) \cong \mathbb{Z}$ which is not weakly second and hence it is not S-weakly second .

(2) As another example of (1), let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ be a ring, $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ an idempotent in R and $M = eR = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ a module over R. We have $S = End_R(M) \cong eRe = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix}$ is a domain implies $ann_S(M) = 0$ is a prime ideal in S but M is not an S-weakly second R-module because if we take $f = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $g = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \in S$ implies $fg(M) = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} bc & bd \\ 0 & 0 \end{pmatrix}, a, b, c, d \in \mathbb{Z} \right\} = \left\{ \begin{pmatrix} abc & abd \\ 0 & 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} ab\mathbb{Z} & ab\mathbb{Z} \\ 0 & 0 \end{pmatrix} \right\}$ but $f(M) = \left\{ \begin{pmatrix} a\mathbb{Z} & a\mathbb{Z} \\ 0 & 0 \end{pmatrix} \right\}$ and $g(M) = \left\{ \begin{pmatrix} b\mathbb{Z} & b\mathbb{Z} \\ 0 & 0 \end{pmatrix} \right\}$ that is neither $fg(M) \neq f(M)$ nor $fg(M) \neq g(M)$.

Corollary 5.10 If M is an S-weakly second R-module then for every proper submodule K of M we have $[K:_S M] = [K:_S \vartheta(M)]$ for each $\vartheta \in S$ with $\vartheta \notin [K:_S M]$.

Proof

Let $\zeta \in [K:_S M]$ then $\zeta(M) \subseteq K$ implies for each $\vartheta \in S$, $\zeta\vartheta(M) \subseteq K$ so $\zeta \in [K:_S \vartheta(M)]$. Conversly, let $\zeta \in [K:_S \vartheta(M)]$ then $\zeta\vartheta(M) \subseteq K$ and so $\zeta\vartheta \in [K:_S M]$. Via Theorem 5.7, $[K:_S N]$ is prime of S and $\vartheta \notin [K:_S M]$ implies that $\zeta \in [K:_S M]$ as required.

Corollary 5.11 If M is an S-weakly second R-module then $ann_S(M) = ann_S(gM)$ for each $g \in S$ with $g \notin ann_S(M)$.

Proof

Directly by Corollary 5.10

Theorem 5.12 See the equivalent below

(1) M is an S-weakly second R-module.

(2) The set $\{[Q:_S M]$ where Q is proper of M $\}$ is a chain of prime ideals of S.

Proof

Similar proof of Theorem 3.7

Theorem 5.13 The next are equivalent

(1) M is an S-weakly second R-module.

(2) $M \neq 0$ and for each ideals I, J of S and K a submodule of M such that $IJM \subseteq K$ implies $IM \subseteq K$ or $JM \subseteq K$.

Proof

(1) $\Rightarrow$ (2) First since M is a weakly second R-module then $M \neq 0$. Let I and J be ideals of S and K a submodule of M. If $M \neq K$ we have either $IJM \nsubseteq K$ and so nothing to prove or $IJM \subseteq K$ it follows $IJ \subseteq [K:_S M]$ and by Theorem 5.7,$[K:_S M]$ is a prime ideal of S so $I \subseteq [K:_S M]$ or $J \subseteq [K:_S M]$ and hence $IM \subseteq K$ or $JM \subseteq K$. In case $M = K$ then the result already is obtained.

(2) $\Rightarrow$ (1) Let $\zeta\vartheta(M) \subseteq K$, where $\zeta, \vartheta \in S$ and K a submodule of M, then $S_\zeta S_\vartheta(M) \subseteq K$ By hypothesis either $S_\zeta(M) \subseteq K$ or $S_\vartheta(M) \subseteq K$ where S_ζ and S_ϑ are the ideals generated by ζ and ϑ respectively in S implies $\zeta(M) \subseteq K$ or $\vartheta(M) \subseteq K$ as dsired.

Corollary 5.14 The next statements are equivalent

(1) M is an S-weakly second R-module.

(2) $M \neq 0$ and for each ideals I and J of S implies $IM = IJM$ or $JM \subseteq IJM$.

Proof

in similar way to the proof of Theorem 5.5 and by Theorem 5.7.

Corollary 5.15 If S is commutative ring we have the equivalent below

(1) M is an S-weakly second R-module.

(2) $M \neq 0$ and for each ideals I and J of S implies $IM = IJM$ or $JM = IJM$.

Proof

It is clear.

Corollary 5.16 The following are balance

(1) M is an S-weakly second of R-module.

(2) $M \neq 0$ and for each ideals I and J of S and K a proper submodule of M and $IJ \subseteq [K:_S M]$ implies $I \subseteq [K:_S M]$ or $J \subseteq [K:_S M]$.

Proof

Directly via Theorem 5.7

Corollary 5.17 The equivalent are equivalent

(1) M is an S-weakly second R-module.

(2) $N \neq 0$ and for each ideals I and J of S and K a proper submodule of $M, IJ \subseteq [K:_S M]$ and $I \nsubseteq [K:_S M]$ implies that $J \subseteq [K:_S M]$.

Proof

Directly via Corollary 5.16

Corollary 5.18 We have the equivalent

(1) M is an S-weakly second R-module.

(2) $M \neq 0$ and for each ideals I and J of S and K a proper submodule of $M, IJ \subseteq [K:_S M]$ and $[K:_S M] \subset I$ implies that $J \subseteq [K:_S M]$.

Proof

Directly via Corollary 5.16

Proposition 5.19 Every weakly second multiplication module is S-weakly second

Proof

Let M be a weakly second multiplication R-module and ζ, $\vartheta \in S$ with $\zeta\vartheta(M) \subseteq K$ for some a submodule K of M. Since M is multiplication then $\zeta\vartheta(M) = \zeta(JM) = J\zeta(M) = IJM$ for ideals I and J of R and hence $IJM \subseteq K$. By Theorem 5.7, either $IM \subseteq K$ or $JM \subseteq K$ it follows $\zeta(M) \subseteq K$ or $\vartheta(M) \subseteq K$ that is M is S-weakly second.

Proposition 5.20 Every weakly second scalar module is S-weakly second

Proof

Let M be a weakly second scalar R-module and ζ, $\vartheta \in S$ with $\zeta\vartheta(M) \subseteq K$ for some K a submodule of M. Since M is scalar then there exist a, $b \in R$ such that $\zeta(m) = am$ and $\vartheta(m) = mb$ for all $m \in M$. Then $K \supseteq \zeta\vartheta(M) = \zeta(Mb) = Mab$ implies $Ma \subseteq K$ or $Mb \subseteq K$ it follows $\zeta(M) \subseteq K$ or $\vartheta(M) \subseteq K$ as desired.

Proposition 5.21 Every summand of S-weakly second module is S-weakly second.

Proof

Let Q be a direct summand of an S-weakly second R-module $\mathfrak{M}$ then $\mathfrak{M} = Q \oplus \mu$ for some submodule μ of M. Let $\zeta, \vartheta \in End(N)$ with $\zeta\vartheta(Q) \subseteq \nabla$ for some ∇ a submodule of Q. We can define $\alpha(n + \hbar) = \zeta(n)$ and $\beta(n + \hbar) = \vartheta(n)$ where $n \in Q$and $\hbar \in \mu$. It is easy to see that α, $\beta \in S$, $\alpha(\mathfrak{M}) = \zeta(Q)$ and $\beta(\mathfrak{M}) = \vartheta(Q)$ implies $\alpha\beta(\mathfrak{M}) = \zeta\vartheta(Q) \subseteq \nabla$ it follows $\alpha(\mathfrak{M}) \subseteq \nabla$ or $\beta(\mathfrak{M}) \subseteq \nabla$ and hence $\zeta(Q) \subseteq \nabla$ or $\vartheta(Q) \subseteq \nabla$ as desired.

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