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On some generalization of normal operators on Hilbert space

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Abstract
In this paper we introduce a new class of operators on Hilbert space. We call the operators in this class, (n,m)- powers operators. We study this class of operators and give some of their basic properties.

Keywords: Hilbert space, normal operator.

حول بعض تعميمات المؤثرات الاعتياديه المعرفه على فضاء هلبرت

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الخلاصه
قدمنا في هذا البحث نوع جديد من المؤثرات على فضاء هلبرت واطلقنا عليه اسم مؤثر القوى- (n,m). لقد قمنا بدراسه هذا النوع من المؤثرات واعطينا بعض خواصها الرئيسيه.

Introduction.

Throughout this paper, let $B(H)$ denotes to the algebra of all bounded linear operates acting on a complex Hilbert space H. In [1] the author introduce the class of n-normal operators as a generalization of the class of normal operators and study some properties of such class for different values of the parameter n. In this paper, we study the bounded operator on the complex Hilbert space H that satisfy the following equation

$$T^n(T^m)^* = (T^m)^* T^n.$$

For some nonnegative integers n, m. Operator T satisfying above equation are said to be (n,m)-powers operator.

Recall that a bounded operator T is n- normal operator if $T^nT^* = T^*T^n$ where n be a nonnegative integer.

In **[1]** prove that a bounded operator T is n-normal operator iff T^n is normal operator where n is nonnegative integar.

The outline of this paper is as follows: Introduction and terminologies are described in the first section. In the first section we introduce the class of (n,m)-powers operators and we develop some basic properties of this class. In the second section we discus the product, tenser product and direct sum of finite numbers of (n,m)-powers oprators on a Hilbert space H. In the tired section we will try to study the relation between the eigenvalues, the eigenvector and the sufficient condition for (n,m)-powers oprators.

1. The basic properties of (*n*, *m*)-powers operators.

In this section, we will study some properties which are applied for (n,m)- powers operator.

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Definition 1.1: Let T be a bounded operator. T is called (n,m)-powers operator iff $T^n(T^m)^* = (T^m)^* T^n$ where *n, m* be two nonnegative integers.

Remark 1.2: It is easily seen that every bounded normal operator is (n,m)-powers operator (where $n = m = 1$). Moreover, one can see that every n-normal operator is (n,m)-powers (where $m = 1$). But the converse is not necessary true in general. The following example shows that T is (n,m)-powers but it is not n-normal.

Example 1.3: Let $T = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$.

Then, $T^* = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$, $T^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, $T^3 = \begin{bmatrix} 8 & 4 \\ 0 & -8 \end{bmatrix}$ and $(T^2)^* = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

This implies that, $T^3(T^2)^* = \begin{bmatrix} 32 & 16 \\ 0 & -32 \end{bmatrix} = (T^2)^* T^3$. Therefore, T is $(3,2)$-powers operator bt it is not 3-normal, since $T^3 T^* = \begin{bmatrix} 20 & -8 \\ -8 & -16 \end{bmatrix} \neq \begin{bmatrix} 16 & 8 \\ 8 & 20 \end{bmatrix} = T^* T^3$.

In the following result, we give the sufficient and necessary condition for the 2×2 matrix to be $(2,2)$-powers oprator.

Example 1.4: Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.Then T is $(2,2)$-powers operator iff $b=c$.

Proof:

T is $(2,2)$-powers operator if and only if $T^2(T^2)^* = (T^2)^* T^2$. Note that,

$T^2 = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix}$. Moreover,

$T^* = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, then $(T^2)^* = \begin{pmatrix} a^2+bc & c(a+d) \\ b(a+d) & d^2+bc \end{pmatrix}$. Note that,

$$(T^2)^* T^2 = \begin{pmatrix} (a^2+bc)^2 + c^2(a+d)^2 & (a+d)(b(a^2+bc)+c(d^2+bc)) \\ (a+d)(b(a^2+bc)+c(d^2+bc)) & b^2(a+d)^2 + (d^2+bc)^2 \end{pmatrix}$$ and

$$T^2(T^2)^* = \begin{pmatrix} (a^2+bc)^2 + b^2(a+d)^2 & (a+d)(c(a^2+bc)+c(d^2+bc)) \\ (a+d)(c(a^2+bc)+b(d^2+bc)) & c^2(a+d)^2 + (d^2+bc)^2 \end{pmatrix}.$$

This implies that T is $(2,2)$-powers operator if and only if $b=c$.

Preposition 1.5: Let $T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ where a,b,c are complex numbers and $n,m \geq 2$. Then T is (n,m)-powers operator iff $b^2(a^{n-1} + a^{n-2}c + \cdots + c^{n-1})(a^{m-1} + a^{m-2}c + \cdots + c^{m-1}) = 0$, $c^m = a^m$ and $c^n = a^n$.

Proof:

Note that,

$$T^n = \begin{pmatrix} a^n & b(a^{n-1} + a^{n-2}c + \cdots + c^{n-1}) \\ o & c^n \end{pmatrix}$$ and

$$\left(T^m\right)^* = \begin{pmatrix} a^m & o \\ b\left(a^{m-1} + a^{m-2}c + \cdots + c^{m-1}\right) & c^m \end{pmatrix}.$$ Hence,

$$T^n\left(T^m\right)^* = \begin{pmatrix} \left[a^n a^m + b^2\left(a^{n-1} + a^{n-2}c + \cdots + c^{n-1}\right)\left(a^{m-1} + a^{m-2}c + \cdots + c^{m-1}\right)\right] & \left[bc^m\left(a^{n-1} + a^{n-2}c + \cdots + c^{n-1}\right)\right] \\ bc^n\left(a^{m-1} + a^{m-2}c + \cdots + c^{m-1}\right) & c^n c^m \end{pmatrix}.$$

$$\left(T^m\right)^* T^n = \begin{pmatrix} a^n a^m & ba^m\left(a^{n-1} + a^{n-2}c + \cdots + c^{n-1}\right) \\ \left[ba^n\left(a^{m-1} + a^{m-2}c + \cdots + c^{m-1}\right)\right] & \left[c^n c^m + b^2\left(a^{n-1} + a^{n-2}c + \cdots + c^{n-1}\right)\left(a^{m-1} + a^{m-2}c + \cdots + c^{m-1}\right)\right] \end{pmatrix}.$$

Therefore, it is simple to see that T is (n,m)-powers operator iff

$b^2\left(a^{n-1} + a^{n-2}c + \cdots + c^{n-1}\right)\left(a^{m-1} + a^{m-2}c + \cdots + c^{m-1}\right) = 0$, $c^m = a^m$ and $c^n = a^n$.

We start this section by the following result.

Preposition 1.6: Let T be a bounded operator. If T is (n,m)-powers operator, then T^{nm} is normal opretor.

Proof:

Assume that T is (n,m)-powers operator, then $T^n\left(T^m\right)^* = \left(T^m\right)^* T^n$. Moreover, it is clear that $\left(T^k\right)^* = \left(T^*\right)^k$ for each nonnegative integer k. Therefore,

$$T^{nm}\left(T^{nm}\right)^* = \left(T^n\right)^m\left(\left(T^m\right)^n\right)^*$$

$$= \underbrace{\left(T^n T^n \cdots T^n\right)}_{m-times}\underbrace{\left(T^m T^m \cdots T^m\right)^*}_{n-times}$$

$$= T^n T^n \cdots T^n\left(T^m\right)^* \cdots \left(T^m\right)^*$$

$$= T^n T^n \cdots \left(T^m\right)^* T^n\left(T^m\right)^* \cdots \left(T^m\right)^*$$

$$\vdots$$

$$= \underbrace{\left(T^m\right)^*\left(T^m\right)^* \cdots \left(T^m\right)^*}_{n-times}\underbrace{\left(T^n T^n \cdots T^n\right)}_{m-times}$$

$$= \underbrace{\left(T^m T^m \cdots T^m\right)^*}_{n-times}\underbrace{\left(T^n T^n \cdots T^n\right)}_{m-times}$$

$$= \left(\left(T^m\right)^n\right)^*\left(T^n\right)^m$$

$$= \left(\left(T^n\right)^m\right)^*\left(T^n\right)^m$$

$$= \left(T^{nm}\right)^* T^{nm}.$$

Hence, T^{nm} is normal operator.

The next theorem study the nature of the class of (n,m)-powers operators on a Hilbert space H.

Theorem 1.7: The class of all (n,m)-powers operators on H is a closed subset of $B(H)$ under scalar multiplication.

Proof:

Put,

$NM(H) = \{\ T \in B(H) : T$ is (n,m)-powers operator on H for some nonnegative integers $n, m\ \}$

Let $T \in NH(H)$, then T is (n,m)-powers and thus $T^n\left(T^m\right)^* = \left(T^m\right)^* T^n$. Let α be a scalar, then

$$(\alpha T)^n\left((\alpha T)^m\right)^* = \alpha^n T^n (\overline{\alpha})^m \left(T^m\right)^* = \alpha^n \overline{\alpha}^m T^n \left(T^m\right)^* = \alpha^n \overline{\alpha}^m \left(T^m\right)^* \left(T^n\right) = \overline{\alpha}^m \left(T^m\right)^* \left(\alpha^n T^n\right).$$
$$= \left((\alpha T)^m\right)^* (\alpha T)^n .$$

Thus $\alpha T \in NM(H)$, therefore the scalar multiplication operation is closed under $NM(H)$.

Now, let T_k be a sequence in $B(H)$ of (n,m)-powers operator converge to T, then after simple computation, one can get that

$$\left\|T^n\left(T^m\right)^* - \left(T^m\right)^* T^n\right\| = \left\|T^n\left(T^m\right)^* - T_k^n\left(T_k^m\right)^* + \left(T_k^m\right)^* T_k^n - \left(T^m\right)^* T^n\right\|$$
$$\leq \left\|T^n\left(T^m\right)^* - T_k^n\left(T_k^m\right)^*\right\| + \left\|\left(T_k^m\right)^* T_k^n - \left(T^m\right)^* T^n\right\| \to 0 \quad \text{as} \quad k \to \infty .$$

This implies that, $T^n\left(T^m\right)^* = \left(T^m\right)^* T^n$, therefore $T \in NM(H)$. Hence $NM(H)$ is closed under scalar multiplication

The following proposition discuss the relation between (n,m)-powers operators and (m,n)-powers operators.

Proposition 1.8: T is (n,m)-powers iff T is (m,n)-powers.

Proof:

Let T be (n,m)-powers, then $T^n\left(T^m\right)^* = \left(T^m\right)^* T^n$. Therefore,

$$T^m\left(T^n\right)^* = \left(\left(T^m\left(T^n\right)^*\right)^*\right)^* = \left(T^n\left(T^m\right)^*\right)^*$$
$$= \left(\left(T^m\right)^* T^n\right)^* = \left(T^n\right)^* T^m .$$

Thus T is (m,n)-powers operator. The converse is similar.

The following results collects some of basic properties of (n,m)-powers operator.

Definition 1.9: If A, B are bounded operator on Hilbert space H**.** Then A, B are unitary equivalent if there is an isomorphism $U : H \to H$ such that $B = UAU^{-1}$. In symbol this is denoted by $A \cong B$. [see 2]

Proposition 1.10: If $T \in B(H)$ is (n,m)- powers , then

1) T^* is (n,m)-powers.

2) If T^{-1} exist then T^{-1} is (n,m)-powers.

3) If $S \in B(H)$ is unitary equivalent to T , then S is (n,m)-powers.

4) If M is a closed subspace of H such that M reduces T , then $(T \mid M)^{nm}$ is normal.

Proof:

Since T is (n,m)-powers, then $T^n\left(T^m\right)^* = \left(T^m\right)^* T^n$**.**

1) Note that, by proposition (1.8)

$$\left(T^*\right)^n\left(\left(T^*\right)^*\right)^m = \left(T^*\right)^n T^m = T^m\left(T^*\right)^n$$
$$= \left(\left(T^*\right)^*\right)^m\left(T^*\right)^n$$

Thus T^* is (n,m)-powers.

2) Note that,

$$\left(T^{-1}\right)^n\left(\left(T^{-1}\right)^*\right)^m = \left(T^n\right)^{-1}\left(\left(T^*\right)^m\right)^{-1} = \left(\left(T^*\right)^m\left(T^n\right)\right)^{-1} = \left(\left(T^n\right)\left(T^m\right)^*\right)^{-1}$$

$$=\left(T^n\left(T^m\right)^*\right)^{-1} = \left(\left(T^m\right)^*\right)^{-1}\left(T^n\right)^{-1} = \left(\left(T^{-1}\right)^*\right)^m\left(T^{-1}\right)^n.$$

Thus T^{-1} is (n,m)-powers operator.

3) Since S is unitary equivalent to T, then $S=UTU^*$. Therefore, $\left(UTU^*\right)^n = UT^nU^*$ (see[3]).
Now,

$$S^n\left(S^m\right)^* = \left(UTU^*\right)^n\left(\left(UTU^*\right)^m\right)^* = UT^nU^*\left(UT^mU^*\right)^*$$
$$= UT^nU^*U\left(T^m\right)^*U^* = UT^n\left(T^m\right)^*U^* = U\left(T^m\right)^*T^nU^*$$
$$= U\left(T^m\right)^*U^*UT^nU^* = \left(\left(T^mU^*\right)^*U^*\right)UT^nU^* = \left(U\left(T^mU^*\right)\right)^*UT^nU^*$$
$$= \left(\left(UTU^*\right)^m\right)^*\left(UTU^*\right)^n$$
$$= \left(S^m\right)^*S^n.$$

Hence S is (n,m)-powers operator.

4) Since T is (n,m)-powers, then by proposition (1.6) T^{nm} is normal. But M reduces T, then $T^{nm}|M$ is normal (see[3]). Moreover, $T^{nm}|M=(T|M)^{nm}$, thus $(T|M)^{nm}$ is normal.

In the following results we study some sufficient condition for (n,m)-powers operator for all n,m.

Proposition 1.11: Let T be (k,m)-powers and $(k+1,m)$-powers operator where k, m are nonnegative integers, then T is $(k+2,m)$-powers. Therefore by induction T is (n,m)-powers operator for all n,m.

Proof:

Since T is (k,m)-powers and T is $(k+1,m)$-powers, then $T^k\left(T^m\right)^* = \left(T^m\right)^*T^k$ and $T^{k+1}\left(T^m\right)^* = \left(T^m\right)^*T^{k+1}$ respectively. Note that,

$$T^{k+2}\left(T^m\right)^* = TT^{k+1}\left(T^m\right)^* = T\left(T^m\right)^*T^{k+1} = T\left(T^m\right)^*T^kT$$
$$= TT^k\left(T^m\right)^*T = T^{k+1}\left(T^m\right)^*T$$
$$= \left(T^m\right)^*T^{k+1}T = \left(T^m\right)^*T^{k+2}$$

Hence by induction T is $(k+2,m)$-powers, this implies that T is (n,m)-powers for all n,m.

The proof of the next result is similar to Proposition (1.11), thus we omitted.

Proposition 1.12: Let T be (n,k)-powers and $(n,k+1)$-powers operator where n, k are nonnegative integers, then T is $(n,k+2)$-powers operator. Therefore by induction T is (n,m)-powers operator for all n, m.

In what follows, we study the sufficient and necessary condition to the weighted shift operator to be (n,m)-powers operator.

Example 1.13: Let T be a weighted shift operator with nonzero weights $\{\alpha_k\}_{k=0}^{\infty}$. Then T is (n,m)-powers operator iff $\overline{\alpha}_{k-1}\ldots\left|\alpha_{k-m}\right|^2\alpha_{k-m+1}\ldots\alpha_{k+n-m-1} = \alpha_k\ldots\left|\alpha_{k+n-1}\right|^2\overline{\alpha}_{k+n-2}\ldots\overline{\alpha}_{k+n-m}$ for $k \geq m$.

Proof : Let $\{e_k\}_0^\infty$ be an orthogonal basis of Hilbert space H. Then $Te_k = \alpha_k e_{k+1}$ and $T^* e_k = \overline{\alpha_{k-1}} e_{k-1}$, note that

$T^n e_k = \alpha_k \dots \alpha_{k+n-1} e_{k+n}$ and $(T^m)^* e_k = (T^*)^m e_k = \overline{\alpha}_{k-1} \dots \overline{\alpha}_{k-m} e_{k-m}$. Hence,

$$T^n (T^m)^* e_k = \overline{\alpha}_{k-1} \dots \overline{\alpha}_{k-m} (T^n e_{k-m}) = \overline{\alpha}_{k-1} \dots \overline{\alpha}_{k-m} (\alpha_{k-m} \dots \alpha_{k-m+n-1})\ e_{k+n-m}$$

$$= \overline{\alpha}_{k-1} \dots |\alpha_{k-m}|^2 \alpha_{k-m+1} \dots \alpha_{k+n-m-1} e_{k+n-m}$$

$$(T^m)^* T^n e_k = \alpha_k \dots \alpha_{k+n-1} (T^m)^* e_{k+n} = \alpha_k \dots \alpha_{k+n-1} (\overline{\alpha_{k+n-1}} \dots \overline{\alpha_{k+n-m}})\ e_{k+n-m}$$

$$= \alpha_k \dots |\alpha_{k+n-1}|^2 \overline{\alpha}_{k+n-2} \dots \overline{\alpha}_{k+n-m} e_{k+n-m}$$

Thus T is (n,m)-powers iff $\alpha_k \dots |\alpha_{k+n-1}|^2 \dots \overline{\alpha}_{k+n-m} = \overline{\alpha}_{k-1} \dots |\alpha_{k-m}|^2 \dots \alpha_{k+n-m-1}$.

2.Some opration on (n,m)-powers operators.

In this section we discuss the product, tenser product and direct sum of finite number of (n,m)-powers operators on a Hilbert space H.

Theorem 2.1: Assume that S commutes with T . If S and T are (n,m)-powers, then (ST) is (n,m)-powers.

Proof:

Since S commute with T , then it is clear that $(ST)^n = S^n T^n$. Moreover, S commutes with T^* and T commutes with S^*. Note that,

$$(ST)^n ((ST)^m)^* = S^n T^n (S^m T^m)^* = S^n T^n (T^m)^* (S^m)^*$$

$$= S^n T^n (T^m)^* (S^m)^* = S^n (T^m)^* T^n (S^m)^* \quad \text{(Since } T \text{ is } (n,m)\text{-powers)}$$

$$= (T^m)^* S^n (S^m)^* T^n = (T^m)^* (S^m)^* S^n T^n \quad \text{(Since } S \text{ is } (n,m)\text{-powers)}$$

$$= ((ST)^m)^* (ST)^n.$$

Thus ST is (n,m)-powers operator

Theorem 2.2: Let $T_1, T_2, \dots, T_p$ be (n,m)-powers operators on $B(H)$, then $(T_1 \oplus T_2 \oplus \dots \oplus T_p)$ is a (n,m)-powers operator.

Proof :

Since each of $T_1, T_2, \dots, T_p$ is (n,m)-normal operator, then $T_i^n (T_i^m)^* = (T_i^m)^* T_i^n$ for all *i=1,2,..,p*, then it is simple to see that

$$(T_1 \oplus T_2 \oplus \dots \oplus T_p)^n ((T_1 \oplus T_2 \oplus \dots \oplus T_p)^m)^* = (T_1^n \oplus T_2^n \oplus \dots \oplus T_p^n)\ (T_1^m \oplus T_2^m \oplus \dots \oplus T_p^m)^*$$

$$= (T_1^n \oplus T_2^n \oplus \dots \oplus T_p^n)((T_1^m)^* \oplus (T_2^m)^* \oplus \dots \oplus (T_p^m)^*)$$

$$= T_1^n (T_1^m)^* \oplus T_2^n (T_2^m)^* \oplus \dots \oplus T_p^n (T_p^m)^*$$

$$= (T_1^m)^* T_1^n \oplus (T_2^m)^* T_2^n \oplus \dots \oplus (T_p^m)^* T_p^n$$

$$= ((T_1^m)^* \oplus (T_2^m)^* \oplus \dots \oplus (T_p^m)^*)(T_1^n \oplus T_2^n \oplus \dots \oplus T_p^n)$$

$$= (T_1^m \oplus T_2^m \oplus \dots \oplus T_p^m)^* (T_1^n \oplus T_2^n \oplus \dots \oplus T_p^n)$$

$$=\left(\left(T_1 \oplus T_2 \oplus \ldots \oplus T_p\right)^m\right)^*\left(T_1 \oplus T_2 \oplus \ldots \oplus T_p\right)^n$$

Thus $\left(T_1 \oplus T_2 \oplus \ldots \oplus T_p\right)$ is (n,m)-normal.

Theorem 2.3: Let $T_1, T_2, \ldots, T_p$ be (n,m)-powers operators on $B(H)$, then $\left(T_1 \otimes T_2 \otimes \ldots \otimes T_p\right)$ is a (n,m)-powers operator.

Proof :

Since each of $T_1, T_2, \ldots, T_p$ is (n,m)-normal operator, then $T_i^n\left(T_i^m\right)^* = \left(T_i^m\right)^* T_i^n$ for all *i=1,2,..,p*. Let $x_1, x_2, \ldots, x_p \in H$, then it is simple to see that

$$\left(T_1 \otimes T_2 \otimes \ldots \otimes T_p\right)^n\left(\left(T_1 \otimes T_2 \otimes \ldots \otimes T_p\right)^m\right)^*\left(x_1 \otimes x_2 \otimes \ldots \otimes x_p\right)$$

$$=\left(T_1^n \otimes T_2^n \otimes \ldots \otimes T_p^n\right)\left(T_1^m \otimes T_2^m \otimes \ldots \otimes T_p^m\right)^*\left(x_1 \otimes x_2 \otimes \ldots \otimes x_p\right)$$

$$=T_1^n\left(T_1^m\right)^* x_1 \otimes T_2^n\left(T_2^m\right)^* x_2 \otimes \ldots \otimes T_p^n\left(T_p^m\right)^* x_p$$

$$=\left(T_1^m\right)^* T_1^n x_1 \otimes \left(T_2^m\right)^* T_2^n x_2 \otimes \ldots \otimes \left(T_p^m\right)^* T_p^n x_p$$

$$=\left(\left(T_1^m\right)^* \oplus \left(T_2^m\right)^* \oplus \ldots \oplus \left(T_p^m\right)^*\right)\left(T_1^n \oplus T_2^n \oplus \ldots \oplus T_p^n\right)\left(x_1 \otimes x_2 \otimes \ldots \otimes x_p\right)$$

$$=\left(T_1^m \otimes T_2^m \otimes \ldots \otimes T_p^m\right)^*\left(T_1^n \otimes T_2^n \otimes \ldots \otimes T_p^n\right)\left(x_1 \otimes x_2 \otimes \ldots \otimes x_p\right)$$

$$=\left(\left(T_1 \otimes T_2 \otimes \ldots \otimes T_p\right)^m\right)^*\left(T_1 \otimes T_2 \otimes \ldots \otimes T_p\right)^n\left(x_1 \otimes x_2 \otimes \ldots \otimes x_p\right)$$

Thus $\left(T_1 \oplus T_2 \oplus \ldots \oplus T_p\right)$ is (n,m)-powers operator.

In the following results we study some sufficient condition for (n,m)- powers operator.

Preposition 2.4: Let $T \in B(H), F = T^n + \left(T^m\right)^*$, and $G = T^n - \left(T^m\right)^*$. Then T is (n,m)-powers operator iff G commutes with F.

Proof :

$FG = GF$ iff $\left(T^n + \left(T^m\right)^*\right)\left(T^n - \left(T^m\right)^*\right) = \left(T^n - \left(T^m\right)^*\right)\left(T^n + \left(T^m\right)^*\right)$

iff $T^{2n} - T^n\left(T^m\right)^* + \left(T^m\right)^* T^n - \left(T^{2m}\right)^* = T^{2n} + T^n\left(T^m\right)^* - \left(T^m\right)^* T^n - \left(T^{2m}\right)^*$

iff $T^n\left(T^m\right)^* = 2\left(T^m\right)^* T^n$

iff $T^n\left(T^m\right)^* = \left(T^m\right)^* T^n$

iff T is (n,m)- powers operator.

Preposition 2.5: Let $T \in B(H), F = T^n + \left(T^m\right)^*$, $G = T^n - \left(T^m\right)^*$ and $B = T^n\left(T^m\right)^*$. If T is (n,m)-powers operator, then B commutes with F and G.

Proof:

Since T is (n,m)-powers, then

$$BF = T^n\left(T^m\right)^*\left(T^n + \left(T^m\right)^*\right)$$

$$= T^n\left(T^m\right)^* T^n + T^n\left(T^m\right)^*\left(T^m\right)^*$$

$$= T^n T^n\left(T^m\right)^* + \left(T^m\right)^* T^n\left(T^m\right)^*$$

$$= \left(T^n + \left(T^m\right)^*\right) T^n\left(T^m\right)^* = FB.$$

Similarly one can show that $BG = GB$.

3. The eigenspace of (n,m)-powers operator.

In this section, we discuss the eigenvalues and the eigenspaces of (n,m)-powers operators. We will try to study the relation between the eigenvalues, the eigenspaces and the sufficient condition for (n,m)-powers operators.

Lemma 3.1 [3]: Let P,Q be two projections on closed subspaces M,N respectively. Then $M \perp N$ iff $PQ=0$.

Lemma 3.2 [3]: If T is normal operator, then $Tx=\lambda x$ iff $T^{*}x=\bar{\lambda}x$.

Lemma 3.3 [3]: If P is projection on closed subspace M of H, then M reduces of T iff $TP=PT$.

Theorem 3.4: Let T be an operator on a finite dimensional Hilbert space H and $\lambda_1,\ldots,\lambda_s$ be corresponding eigenvalues of T such that $\lambda_i^{nm} \neq \lambda_j^{nm}, i \neq j$, $i, j=1,\ldots,s$. If $M_1,\ldots,M_s$ be the corresponding eigenspaces and $P_1,\ldots,P_s$ be the projections on $M_1,\ldots,M_s$ respectively, then M_i's are pairwise orthogonal and they span H iff T is (n,m)-powers operator.

Proof:

Assume that M_i's are pairwise orthogonal and they span H. Then for $x \in H$,

$x = x_1 + x_2 + \ldots + x_s$; $x_i \in M_i$, we get

$$T^n x = T^n x_1 + T^n x_2 + \ldots + T^n x_s = \lambda_1^n x_1 + \ldots + \lambda_s^n x_s.$$

Since P_i's are projections on eigenspaces M_i's which are pairwise orthogonal, then we have $P_i x = x_i$. Hence for every $x \in H$,

$x = I\ x = x_1 + x_2 + \ldots + x_s = P_1 x + \ldots + P_s x = (P_1 + \ldots + P_s)x$.

Thus $I = \sum_{i=1}^{s} P_i$. Since for all $x \in H$, we have

$T^n x = \lambda_1^n x_1 + \ldots + \lambda_s^n x_s == \lambda_1^n P_1 x + \ldots + \lambda_s^n P_s x = (\lambda_1^n P_1 + \ldots + \lambda_s^n P_s)x$. So that $T^n = \sum_{i=1}^{s} \lambda_i^n P_i$. Hence $(T^m)^* = \sum_{i=1}^{s} \bar{\lambda}_i^m P_i$. In addition that since M_i's are pairwise orthogonal, then $P_i P_j = \begin{cases} P_i & if\ \ i=j \\ 0 & if\ \ i \neq j \end{cases}$.

Thus $T^n (T^m)^* = \lambda_1 \bar{\lambda}_1 P_1 + \ldots + \lambda_s \bar{\lambda}_s P_s = \bar{\lambda}_1 \lambda_1 P_1 + \ldots + \bar{\lambda}_s \lambda_s P_s = (T^m)^* T^n$. Therefore T is (n,m)- powers.

Conversely, suppose T is an (n,m)-powers operator, then by proposition (1.6) T^{nm} is normal operator. We claim that M_i's are pairwise orthogonal.

Let x_i, x_j be two vectors in M_i, M_j respectively such that $i \neq j$ where $i, j=1,\ldots,s$. Note that, $T^{nm} x_i = \lambda_i^{nm} x_i$ and $T^{nm} x_j = \lambda_j^{nm} x_j$. Thus,

$$\lambda_i^{nm} \langle x_i, x_j \rangle = \langle \lambda_i^{nm} x_i, x_j \rangle = \langle T^{nm} x_i, x_j \rangle$$

$$= \langle x_i, (T^*)^{nm} x_j \rangle$$

$$= \langle x_i, \overline{\lambda_j^{nm}} x_j \rangle = \lambda_j^{nm} \langle x_i, x_j \rangle.$$

So that $(\lambda_i^{nm} - \lambda_j^{nm})\langle x_i, x_j \rangle = 0$. But $\lambda_i^{nm} \neq \lambda_j^{nm}$, then $\langle x_i, x_j \rangle = 0$ for all *i, j=1,...,s*. This shows that M_i's are pairwise orthogonal. Let $M = M_1 + \ldots + M_s$, then it is clear that M is a closed subspace of H. Let P be an associated projection onto M, then $P = P_1 + \ldots + P_s$. Since T^{nm} is normal, then each M_i reduces T^{nm} (see[3]). It follows by lemma (3.3) that $T^{nm}P = PT^{nm}$. Consequently $M^{\perp}$ is invariant under T^{nm}. Suppose that $M^{\perp} \neq \{0\}$. Let $T_1 = T^{nm} \mid M^{\perp}$. Then T_1 is an operator on non-trivial finite dimension complex Hilbert space $M^{\perp}$ with empty point spectrum which is impossible. Therefore

$M^{\perp} = \{0\}$. $H = M \oplus M^{\perp} = H$.

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