DOI: 10.24996/ijs.2019.60.7.18

ISSN: 0067-2904

On Jacobson – Small Submodules

Ali Kabban , Wasan Khalid
Department of mathematics, College of Science, University of Baghdad, Baghdad, Iraq

Abstract
Let R be an associative ring with identity and let M be a unitary left R–module . As a generalization of small submodule , we introduce Jacobson–small submodule (briefly J–small submodule) . We state the main properties of J–small submodules and supplying examples and remarks for this concept . Several properties of these submodules are given . Also we introduce Jacobson–hollow modules (briefly J–hollow) . We give a characterization of J–hollow modules and gives conditions under which the direct sum of J–hollow modules is J–hollow . We define J–supplemented modules and some types of modules that are related to J–supplemented modules and introduce properties of this types of modules . Also we discuss the relation between them with examples and remarks are needed in our work.

Keywords : J–small submodules , J–hollow modules , J–supplemented modules , weakly J–supplemented modules.

حول المقاسات الجزئية الصغيرة من النمط – J

علي كبان ، وسن خالد
قسم الرياضيات ، كلية العلوم ، جامعة بغداد، بغداد، العراق

الخلاصة
لتكن R حلقة تجميعية ذات عنصر محايد وليكن M مقاساً ايسر معرف عليها . كتعميم على المقاس الجزئي الصغير, نحن نقدم المقاس الجزئي الصغير من النمط–J . نذكر الخصائص الرئيسية للمقاسات الجزئية الصغيرة من النمط–J ونقدم امثلة وملاحظات لهذا المفهوم . يتم اعطاء العديد من خصائص هذه المقاسات الجزئية . وايضاً نقدم المقاسات المجوفة من النمط–J . نعطي تعميمات حول المقاسات المجوفة من النمط–J ونعطي الشروط التي بموجبها جمع المقاسات المجوفة من النمط–J يعطينا مقاس مجوف من النمط–J . نعرف المقاسات المكملة من النمط–J وبعض انواع المقاسات المرتبطة بالمقاسات المكملة من النمط–J ونقدم خصائص هذه الانواع من المقاسات . وايضاً نناقش العلاقة بينهما مع الامثلة والملاحظات التي نحتاجها في عملنا .

1. Introduction

Throughout this paper , all rings are associative with unity and modules are unital left R–modules , where R denotes such a ring and M denotes such a module. A submodule N of M is called a small submodule of M if whenever N+ K= M for some submodule K of M , we have M = K , and in this case we write N << M, See [1]. A nonzero module M is called hollow module , if every proper submodule of M is small in M . See [1] . It is known that the Jacobson radical of M denoted by J (M) is the sum of all small submodules of M . In fact we use J (M) to introduce a generalization of small submodules , A submodule N of M is called J–small submodule of M (denoted by $N \ll_J M$) if

*Email: alikuban5@gmail.com

whenever M = N + K , with $J(\frac{M}{K}) = \frac{M}{K}$ implies M = K . Also , we define J–hollow modules as generalizations hollow modules A non - zero R–module M is called J–hollow module if every proper submodule of M is J–small submodule of M . And give the basic properties of this concept and prove a characterization of J–hollow modules and give certain conditions under which the direct sum of J–hollow modules is J–hollow . Let M be a module . For N , L ⊆ M , N is a supplement of L in M if N is minimal with respect to M = N + L . Equivalently , M = N + L with N ∩ L << N . See [2].

M is called a weakly supplemented module if for each submodule N of M there exists a submodule K of M such that M = N + K and N ∩ K << M . See [3] . A module M is called supplemented if any submodule N of M has a supplement in M . See [3] . As a generalization of supplemented module . We define J–supplemented module , let M be any R–module and let V,U be submodule of M , V is J–supplement of U in M if the V + U = M , and $V \cap U \ll_J V$ and M is called J–supplemented if every submodule of M has J–supplement submodule . Finally as generalizations of J–supplemented module . We define weakly J–supplemented module , a submodule V is weak J–supplement of U in M if M = U + V and $U \cap V \ll_J M$, and M is called weakly J–supplemented if every submodule of M has J–supplement in M .

2. J–Small submodules and J–Hollow Modules .

In this section , we introduce J–small submodules as a generalization of small submodules , we illustrate this concept by examples and remarks and give the properties of J–small submodules . As a generalization hollow modules and by using the concept of J–small submodule we introduce J–hollow module .

Definition (2.1) : Let M be any R–module a submodule N of M is called **Jacobson–small** (for short J–small, denoted by $N \ll_J M$) if whenever M = N+K , K ⊆ M ,such that $J(\frac{M}{K}) = \frac{M}{K}$ implies M = K

Before we prove some properties of J–small submodule , we need the following .

Proposition (2.2) : Let A , B be submodule of an R–module M if A ⊆ B ⊆ M and $J(\frac{M}{A}) = \frac{M}{A}$, then $J(\frac{M}{B}) = \frac{M}{B}$

Proof: Let $f : \frac{M}{A} \longrightarrow \frac{M}{B}$ be an epimorphism , be defined by $f(m + A) = m + B$, since $f(J(\frac{M}{A})) \subseteq J(\frac{M}{B})$. Hence $f(\frac{M}{A}) = \frac{M}{B} \subseteq J(\frac{M}{B})$. Therefor $J(\frac{M}{B}) = \frac{M}{B}$.

Corollary (2.3) : Let M be any R–module , and let A , B be submodule of M . If $J(\frac{M}{A}) = \frac{M}{A}$, then $J(\frac{M}{A+B}) = \frac{M}{A+B}$.

Examples (2.4) :

1) It is clear that every small submodule of any R–module M is J–small . But the convers is not true in general . For example $\mathbb{Z}_6$ as $\mathbb{Z}$–module , let $A = \{\bar{0}, \bar{3}\}$, $B = \{\bar{0}, \bar{2}, \bar{4}\}$, $A + B = \mathbb{Z}_6$ with $J(\frac{\mathbb{Z}_6}{B}) \neq \frac{\mathbb{Z}_6}{B}$. Thus A is J–small submodule of $\mathbb{Z}_6$ but not small in $\mathbb{Z}_6$.

2) $\mathbb{Z}_4$ as $\mathbb{Z}$–module , by (1) , $\{\bar{0}\}$ and $\{\bar{0}, \bar{2}\}$ are J–small in $\mathbb{Z}_4$

3) Consider $M = \mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$– module Since $\frac{M}{\mathbb{Z}} \cong \mathbb{Z}_{p^\infty}$, then $J(\frac{M}{\mathbb{Z}}) \cong J(\mathbb{Z}_{p^\infty}) = \mathbb{Z}_{p^\infty} \cong \frac{M}{\mathbb{Z}}$. $J(\frac{M}{\mathbb{Z}}) = \frac{M}{\mathbb{Z}}$, but $M \neq \mathbb{Z}$. This mean that $\mathbb{Z}_{p^\infty}$ is not J–small in M .

Proposition (2.5) : Let M be an R–module . If J(M) = M and A ⊆ M . Then $A \ll_J M$ if and only if $A \ll M$.

Proof: ⟹ Let L ⊆ M and let A + K = M , to prove K = M , we claim that $J(\frac{M}{K}) = \frac{M}{K}$, let $f : M \longrightarrow \frac{M}{K}$ be the natural epimorphism since $f(J(M)) \subseteq J(\frac{M}{K})$, and J (M) = M , then $f(M) \subseteq J(\frac{M}{K})$, therefore $\frac{M}{K} \subseteq J(\frac{M}{K})$ and $J(\frac{M}{K}) = \frac{M}{K}$, since $A \ll_J M$ then K = M , hence $A \ll M$.

⟸ Clearly by (Examples 2.4.(1)) .

Note: By Proposition (2.5) we can show easily that the infinite sum of J–small is not J–small .

Proposition (2.6) : Let M be any R–module .

1) Let A ⊆ B ⊆ M . Then $B \ll_J M$ if and only if $\frac{B}{A} \ll_J \frac{M}{A}$ and $A \ll_J M$.

2) Let A and B be a submodules of M . Then $A + B \ll_J M$ if and only if $A \ll_J M$ and $B \ll_J M$.

3) Let $A_1 , A_2 , \dots , A_n \leq M$. Then $\sum_{i=1}^{n} A_i \ll_J M$ if and only if $A_i \ll_J M$, $\forall$ i = 1 , 2 , … , n

4) Let $A \subseteq B$ be submodules of M. If $A \ll_J B$, then $A \ll_J M$.
5) Let $f: M \longrightarrow N$ be a homomorphism. If $A \ll_J M$, then $f(A) \ll_J N$.
6) Let $M = M_1 \oplus M_2$ be R–module and let $A_1 \subseteq M_1$ and $A_2 \subseteq M_2$. Then $A_1 \oplus A_2 \ll_J M_1 \oplus M_2$ if and only if $A_1 \ll_J M_1$ and $A_2 \ll_J M_2$.

Proof:(1):$\Longrightarrow$ Let $B \ll_J M$. To prove $\frac{B}{A} \ll_J \frac{M}{A}$, let $L \subseteq M$ and $\frac{L}{A} \subseteq \frac{M}{A}$, suppose that $\frac{B}{A} + \frac{L}{A} = \frac{M}{A}$ and $J(\frac{M}{L}) = \frac{M}{L}$ to prove $\frac{M}{A} = \frac{L}{A}$, since $\frac{B+L}{A} = \frac{M}{A}$ then $B + L = M$ and $B \ll_J M$, hence $M = L$ and $\frac{M}{A} = \frac{L}{A}$. Therefore $\frac{B}{A} \ll_J \frac{M}{A}$, to prove $A \ll_J M$, let $L \subseteq M$ suppose that $A + L = M$ and $J(\frac{M}{L}) = \frac{M}{L}$, since $B \ll_J M$ and $A \subseteq B$ then $B + L = M$ and since $J(\frac{M}{L}) = \frac{M}{L}$, thus $M = L$, so $A \ll_J M$.

$\Longleftarrow$ suppose that $\frac{B}{A} \ll_J \frac{M}{A}$ and $A \ll_J M$ to prove $B \ll_J M$. Let $L \subseteq M$ suppose that $B + L = M$ and $J(\frac{M}{L}) = \frac{M}{L}$ to prove $M = L$, $\frac{M}{A} = \frac{B+L}{A}$ since $A \subseteq B$ then $A + B = B$, so $\frac{B+L}{A} = \frac{B+L+A}{A} = \frac{B}{A} + \frac{L+A}{A} = \frac{M}{A}$, to prove $J(\frac{\frac{M}{A}}{\frac{L+A}{A}}) = J(\frac{M}{L+A}) = \frac{M}{L+A}$, since $J(\frac{M}{L}) = \frac{M}{L}$, then by Corollary (2.3) $J(\frac{M}{L+A}) = \frac{M}{L+A}$, since $\frac{B}{A} \ll_J \frac{M}{A}$. Then $\frac{L+A}{A} = \frac{M}{A}$ thus $A+L = M$, since $A \ll_J M$, so $M = L$, therefore $B \ll_J M$.

(2) $\Longrightarrow$ Let $A + B \ll_J M$ to show $A \ll_J M$ and $B \ll_J M$, let $A + C = M$ and $J(\frac{M}{C}) = \frac{M}{C}$ to prove $M = C$, since $A + B \ll_J M$ then $(A + B) + C = M$, and since $J(\frac{M}{C}) = \frac{M}{C}$, thus $M = C$, then $A \ll_J M$, and similarity $B \ll_J M$

$\Longleftarrow$ Let $A \ll_J M$ and $B \ll_J M$ to show $A + B \ll_J M$, let $A + B + C = M$, and $J(\frac{M}{C}) = \frac{M}{C}$ to prove $M = C$ since $A \ll_J M$, then $B + C = M$, since $J(\frac{M}{C}) = \frac{M}{C}$, then by Corollary (2.3) $J(\frac{M}{B+C}) = \frac{M}{B+C}$, thus $B + C = M$, and since $B \ll_J M$, and $J(\frac{M}{C}) = \frac{M}{C}$, then $M = C$ and $A + B \ll_J M$.

(3) By induction. Let $A_1 + A_2 + B = M$, with $J(\frac{M}{B}) = \frac{M}{B}$, by Corollary (2.3) $J(\frac{M}{A_2+B}) = \frac{M}{A_2+B}$, since $A_1 \ll_J M$, we get $M = A_2 + B$, since $A_2 \ll_J M$, thus $M = B$. Suppose the relate is true for all $K \in N$. Let $A_1+A_2 + \ldots + A_n+B = M$ with $J(\frac{M}{B}) = \frac{M}{B}$. Then $(A_1 + A_2 + \ldots + A_{n-1}) + A_n+B = M$, since $A_n \ll_J M$, and $J(\frac{M}{A_1 + A_2 + \ldots + A_{n-1}+B}) = \frac{M}{A_1 + A_2 + \ldots + A_{n-1}+B}$, hence we get $A_1+A_2+ \ldots + A_{n-1}+ B = M$. Continue until we get $A_1 + B = M$, $A_1 \ll_J M$ thus $M = B$

(4) Suppose that $A + C = M$ and $J(\frac{M}{C}) = \frac{M}{C}$. To prove $M = C$, then $B \cap (A + C) = M \cap B$ and $B \cap (A + C) = B$ (by modular law), $A + (B \cap C) = B$, to prove $J(\frac{B}{B \cap C}) = \frac{B}{B \cap C}$, by the (Second isomorphism) $\frac{B}{B \cap C} \cong \frac{B+C}{C} \cong \frac{M}{C}$. But $J(\frac{M}{C}) = \frac{M}{C}$, hence $J(\frac{B}{B \cap C}) = \frac{B}{B \cap C}$ and $A \ll_J B$, then $B \cap C = B$, so $B \subseteq C$ and $A \subseteq C$, but $A + C = M$, then $M = C$, thus $A \ll_J M$.

(5) By the (First isomorphism) $\frac{M}{\mathrm{Ker}\, f} \cong f(M)$, but $M = M + \mathrm{Ker}\, f$, $\frac{M+\mathrm{Ker}\, f}{\mathrm{Ker}\, f} \cong f(M)$, but $A \subseteq M$, then $\frac{A+\mathrm{Ker}\, f}{\mathrm{Ker}\, f} \cong f(A)$, and $A \subseteq A + \mathrm{Ker}\, f$, since $A \ll_J M$. Then $A + \mathrm{Ker}\, f \ll_J M$, by Proposition (2.6.(1)), $\frac{A+\mathrm{Ker}\, f}{\mathrm{Ker}\, f} \ll_J \frac{M}{\mathrm{Ker}\, f}$ since $f(A) \cong \frac{A+\mathrm{Ker}\, f}{\mathrm{Ker}\, f} \ll_J \frac{M}{\mathrm{Ker}\, f} \cong f(M)$, then $f(A) \ll_J f(M) \subseteq N$, by Proposition (2.6.(4)), $f(A) \ll_J N$.

(6) $\Longrightarrow$ Let $A_1 \oplus A_2 \ll_J M_1 \oplus M_2$, to show $A_1 \ll_J M_1$ and $A_2 \ll_J M_2$ Let $\pi_1: M_1 \oplus M_2 \longrightarrow M_1$ the projection map define as follows, $\pi_1(m_1 + m_2) = m_1$, for all $m_1 + m_2 \in M_1 \oplus M_2$. Since $A_1 \oplus A_2 \ll_J M_1 \oplus M_2$, then by Proposition (2.6.(5)), $\pi_1(A_1 \oplus A_2) \ll_J \pi_1(M_1 \oplus M_2)$, by definition of π_1. We obtain, $A_1 \ll_J M_1$, and similarity $A_2 \ll_J M_2$.

$\Longleftarrow$Let $A_1 \ll_J M_1$ and $A_2 \ll_J M_2$. To show $A_1 \oplus A_2 \ll_J M_1 \oplus M_2$, $A_1 \ll_J M_1 \subseteq M$, and $A_2 \ll_J M_2 \subseteq M$, then by Proposition (2.6.(4)), $A_1 \ll_J M$ and $A_2 \ll_J M$. By Proposition (2.6.(2)), $A_1 \oplus A_2 \ll_J M = M_1 \oplus M_2$.

Proposition (2.7): Let M be an R–module and $A \subseteq B \subseteq M$. If B is a direct summand of M and $A \ll_J M$, then $A \ll_J B$.

Proof: Let A + L = B , and $J(\frac{B}{L}) = \frac{B}{L}$. To prove B = L . Suppose that $M = B \oplus B_1$, $M = A+L+B_1$, then by Corollary (2.3) , $J(\frac{M}{L+B_1}) = \frac{M}{L+B_1}$ and $A \ll_J M$, then $M = L + B_1$ but $L \cap B_1 = 0$, then $M = L \oplus B_1$, but $M = B \oplus B_1$ and $L \subseteq B$, then B = L , and hence $A \ll_J B$.

Proposition (2.8) : Let M be an R–module and let A , B and C are submodules of M with $A \subseteq B \subseteq C \subseteq M$, if $B \ll_J C$ then $A \ll_J M$.

Proof : Suppose that A + K = M and $J(\frac{M}{K}) = \frac{M}{K}$, to prove M = K since $C \subseteq M$, hence $C = M \cap C = (A+K) \cap C = A+(K \cap C)$, (by modular law), since $A \subseteq B$, $C = B+ (K \cap C)$, to prove $J(\frac{C}{K\cap C}) = \frac{C}{K\cap C}$, by the (Second isomorphism), $\frac{C}{K\cap C} \cong \frac{C+K}{K} \cong \frac{M}{K}$, but $J(\frac{M}{K}) = \frac{M}{K}$, then $J(\frac{C}{K\cap C}) = \frac{C}{K\cap C}$, since $B \ll_J C$, then $C = K \cap C$ and $C \subseteq K$ but $A \subseteq C$ hence $A \subseteq K$ then A + K = K , since A + K = M , then K = M and hence $A \ll_J M$.

Note : The converse of Proposition (2.8) is not true in general . As the following example shows . $\mathbb{Z} \subseteq \mathbb{Z}_{p^\infty} \subseteq \mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$ it is clear that $\mathbb{Z} \ll_J \mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$, but $\mathbb{Z}_{p^\infty}$ is not J– small in $\mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$.

Definition (2.9) : A non-zero R–module M is called Jacobson–hollow module (for short J–hollow) if every proper submodule of M is a J–small submodule of M .

Examples and Remarks (2.10) :

1) It is clear that every hollow module is J–hollow module . But the converse in general is not true . For example $\mathbb{Z}_6$ as $\mathbb{Z}$–module . It is clear every proper submodule of $\mathbb{Z}_6$ is J– small , but not small , hence $\mathbb{Z}_6$ as $\mathbb{Z}$–module is J–hollow , but not hollow .

2) $\mathbb{Z}_4$ as $\mathbb{Z}$–module is J–hollow .

3) Consider $M = \mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$–module is not J–hollow . Since $\mathbb{Z}_{p^\infty}$ proper submodule of M but $\mathbb{Z}_{p^\infty}$ is not J–small of M .

4) Every simple module is a J–hollow . For example $\mathbb{Z}_2$ as $\mathbb{Z}$–module .

Proposition (2.11) : A non-zero epimorphic image of J–hollow module is J–hollow .

Proof : Let $f : M \longrightarrow N$ be an epimorphism , and let M be J–hollow module , with $K \subsetneq N$ to show $K \ll_J N$, since $K \subsetneq N$ then $f^{-1}(k) \subsetneq M$. If $f^{-1}(k) = M$ then $K = f(M) = N$, hence K = N this is a contradiction and since M is J–hollow , therefore $f^{-1}(k) \ll_J M$, and by Proposition (2.6.(5)) , $f(f^{-1}(k)) \ll_J N$, then $K \ll_J N$.

Corollary (2.12) : Let M be an R–module and $A \subseteq M$ if M is J–hollow then $\frac{M}{A}$ is J–hollow .

Proof : Let $f : M \longrightarrow \frac{M}{A}$ be the natural epimorphism , and let M be J–hollow . By Proposition (2.11) we get $\frac{M}{A}$ is J–hollow .

Recall that a submodule N of M is called fully invariant if $f(N) \subseteq N$, for each $f \in End(M)$, and M is called duo module if every submodule of M is fully invariant [4] .

Proposition (2.13) : Let $M = M_1 \oplus M_2$, M is duo module then M is J–hollow if and only if M_1 and M_2 are J–hollow . Provided $A \cap M_i \neq M_i$ for all i = 1 , 2.

Proof : $\Longrightarrow$ Let M is J–hollow and $A_1 \oplus A_2 \subsetneq M_1 \oplus M_2$,with $A_1 \subsetneq M_1$ and $A_2 \subsetneq M_2$, and $A_1 \oplus A_2 \ll_J M_1 \oplus M_2 = M$ to show M_1 is J–hollow . Let $\pi_1 : M_1 \oplus M_2 \longrightarrow M_1$ the projection map , define as follows , $\pi_1(m_1 + m_2) = m_1$, for all $m_1 + m_2 \in M_1 \oplus M_2$, since $A_1 \oplus A_2 \ll_J M_1 \oplus M_2$, then by Proposition (2.6.(5)) , $\pi_1(A_1 \oplus A_2) \ll_J \pi_1(M_1 \oplus M_2)$, then , $A_1 \ll_J M_1$, thus M_1 is J–hollow and similarity M_2 is J–hollow .

$\Longleftarrow$ Let M_1 and M_2 are J–hollow to show $M = M_1 \oplus M_2$ is J–hollow . Let $A_1 \subsetneq M_1$ and $A_1 \ll_J M_1$, let $A_2 \subsetneq M_2$ and $A_2 \ll_J M_2$, to show $A_1 \oplus A_2 \ll_J M_1 \oplus M_2$, since $A_1 \ll_J M_1 \subseteq M$, and $A_2 \ll_J M_2 \subseteq M$ then by Proposition(2.6.(4)) $A_1 \ll_J M$ and $A_2 \ll_J M$. By Proposition(2.6.(2)) $A_1 \oplus A_2 \ll_J M = M_1 \oplus M_2$.

3. J–Supplemented Modules and Weakly J–Supplemented Modules

In this section , we give some properties of Jacobson–supplement submodules and weak Jacobson – supplement submodule . There are also some relations and generalizations between supplement submodule and Jacobson –supplement submodules are also between weak supplement submodule and weak Jacobson–supplement submodule.

Definition (3.1): Let M be any R–module and let N , K be submodules of M . N is called **Jacobson – supplement** of K in M (for short J–supplement) if the N + K = M , and $N \cap K \ll_J N$. If every submodule of M has J–supplement then M is called **J–supplemented** .

It easy to prove the following

Remark (3.2): Let M be any R–module and let N, K be a submodules of M . N is J–supplement of K in M if and only if for each L ⊆ N with $J(\frac{N}{L}) = \frac{N}{L}$, and M = L + K implies L = N .

Examples and Remarks (3.3):

1) Every semisimple module is J–supplemented . In particular. $\mathbb{Z}_6$ as $\mathbb{Z}$–module is J–supplemented .

2) $\mathbb{Q}$ as $\mathbb{Z}$–module is not J–supplemented . By Proposition (2.5)

3) Every supplemented is J–supplemented but the converse is not true . See the following example . $\mathbb{Z}$ as $\mathbb{Z}$–module is J–supplemented but not supplemented . Let n , $m \in$ N , a submodule ($m\mathbb{Z}$) has no supplement in $\mathbb{Z}$ because $m\mathbb{Z} + n\mathbb{Z} = \mathbb{Z}$ and g.c.d (m , n) = 1 , and $m\mathbb{Z} \cap n\mathbb{Z} = (m\,n)\mathbb{Z}$ not small in $m\mathbb{Z}$. And $n\mathbb{Z}$ is J–supplemented of $m\mathbb{Z}$ since $m\mathbb{Z} + n\mathbb{Z} = \mathbb{Z}$, and $m\mathbb{Z} \cap n\mathbb{Z} = (m\,n)\mathbb{Z}$, $(m\,n)\mathbb{Z} + k\mathbb{Z} = \mathbb{Z}$, and for each $k\mathbb{Z} \subseteq \mathbb{Z}$, $J(\frac{\mathbb{Z}}{k\mathbb{Z}}) \neq \frac{\mathbb{Z}}{k\mathbb{Z}}$, thus $n\mathbb{Z}$ is J–supplemented of $m\mathbb{Z}$ in $\mathbb{Z}$.

Proposition (3.4) : Let M be a J–supplemented module and let N ⊆ M then $\frac{M}{N}$ is a J–supplemented .

Proof : Let $\frac{K}{N} \subseteq \frac{M}{N}$ to prove $\frac{K}{N}$ has J–supplement in $\frac{M}{N}$, K ⊆ M , and M is J–supplemented , then there exists L ⊆ M such that M = K + L , and $K \cap L \ll_J L$, now $\frac{M}{N} = \frac{K+L}{N} = \frac{K}{N} + \frac{L+N}{N}$, to prove $\frac{K}{N} \cap \frac{L+N}{N} \ll_J \frac{L+N}{N}$, let $(\frac{K}{N} \cap \frac{L+N}{N}) + \frac{V}{N} = \frac{L+N}{N}$ with $J(\frac{L+N}{V}) = \frac{L+N}{V}$ to prove $\frac{V}{N} = \frac{L+N}{N}$, $\frac{K \cap (L+N)}{N} = \frac{N+(K\cap L)}{N}$, then $\frac{N+(K\cap L)}{N} + \frac{V}{N} = \frac{L+N}{N}$, and N + (K ∩ L) + V = L + N , and N ⊆ V then (K ∩ L) + V = L + N , and $J(\frac{L+N}{V}) = \frac{L+N}{V}$, but $K \cap L \ll_J L \subseteq L + N$ and by Proposition (2.6.(4)), $K \cap L \ll_J L + N$, thus V = L + N and $\frac{V}{N} = \frac{L+N}{N}$.

Proposition (3.5): Let M_1 , U ⊆ M , and let M_1 be J–supplemented module . If M_1+ U has a J–supplement in M then so does U .

Proof: Since M_1+ U has a J–supplement in M , then there exists X ⊆ M , such that X + (M_1+ U) = M , and $X \cap (M_1 + U) \ll_J X$. Since M_1 is J–supplemented module , then there exists $Y \subseteq M_1$ such that $(X + U) \cap M_1 + Y = M_1$ and $(X + U) \cap Y \ll_J Y$. Thus we have X + U + Y = M , and $(X + U) \cap Y \ll_J Y$, that is Y is a J–supplement of X + U in M . Next , we will show that X + Y is a J–supplement of U in M , it is clear that (X + Y) + U = M , so it suffices to show that $(X + Y) \cap U \ll_J X + Y$ since $Y + U \subseteq M_1 + U$, by Proposition (2.6.(4)) , $X \cap (Y + U) \subseteq X \cap (M_1 + U) \ll_J X$. Thus by Proposition (2.6. (5)) , $(X + Y) \cap U \subseteq X \cap (Y + U) + Y \cap (X + U) \ll_J X + Y$.

Proposition (3.6): Let $M = M_1 \oplus M_2$, then M_1 and M_2 are J–supplemented module if and only if M is J–supplemented module.

Proof : ⟹ Let U ⊆ M since $M_1 + M_2 + U = M$, trivially has a J–supplement in M . By Proposition (3.5) then M_2 + U has a J–supplement in M and by Proposition (3.5) again U has a J–supplement in M, so M is a J–supplemented module .

⟸ $M_2 \cong \frac{M}{M_1}$,since M is a J–supplemented module and by Proposition (3.4) $\frac{M}{M_1}$ is a J–supplemented module . Thus M_2 is a J–supplemented module . Similarity from M_1 is a J–supplemented module.

Corollary (3.7): Let $M = M_1 \oplus M_2$ be a duo module , N and L are submodule of M_1 , if N is a J–supplement of L in M_1 , then $N \oplus M_2$ is J–supplement of L in M .

Proof : Let N be J–supplement of L in M_1 , then $M_1 = N + L$ and $N \cap L \ll_J N$ since $M = M_1 \oplus M_2$, then $M = (N + L) \oplus M_2$, hence $M = L + (N \oplus M_2)$, but $(N \oplus M_2) \cap L = (N \oplus M_2) \cap M_1 \cap L =$

$N \cap L \ll_J N$. And by Proposition (2.6.(4)) , then $N \cap L \ll_J N \oplus M_2$, hence $N \oplus M_2$ is a J–supplement of L in M .

Proposition (3.8): Let M be any R–module and let V, U be submodule of M , V is a J – supplement of U in M , then $\frac{V+L}{L}$ is J–supplement of $\frac{U}{L}$ in $\frac{M}{L}$, for $L \subseteq U$.

Proof: Since V is a J–supplement of U in M. Then M = U + V and $U \cap V \ll_J M$ for $L \subseteq U$ we have $U \cap (V + L) = (U \cap V) + L$ (by modular law) , and $\frac{U}{L} \cap (\frac{V+L}{L}) = \frac{(U \cap V)+L}{L}$, since $U \cap V \ll_J V$, it follows that $\frac{(U \cap V)+L}{L} \ll_J \frac{V+L}{L}$. Now $\frac{M}{L} = \frac{U+V}{L} = \frac{U}{L} + \frac{V+L}{L}$. Therefor $\frac{V+L}{L}$ is a J–supplement of $\frac{U}{L}$ in $\frac{M}{L}$.

Proposition (3.9): Let M be an R–module . If A has a J–supplement submodule in M, Then $\frac{A}{N}$ has a J–supplement submodule in $\frac{M}{N}$, where N is submodule of A .

Proof : Since A has J–supplement in M then there exists submodule K of M , such that A + K = M , and $A \cap K \ll_J A$. Now we have $\frac{A}{N} + \frac{K+N}{N} = \frac{M}{N}$, to show $\frac{A}{N} \cap \frac{K+N}{N} \ll_J \frac{A}{N}$, $\frac{A}{N} \cap \frac{K+N}{N} = \frac{A \cap (K+N)}{N} = \frac{(A \cap K)+N}{N}$ (by modular law) . Let $\frac{(A \cap K)+N}{N} + \frac{L}{N} = \frac{A}{N}$, with $J(\frac{A}{L}) = \frac{A}{L}$ .To prove $\frac{L}{N} = \frac{A}{N}$, where $L \subseteq A$ and $N \subseteq L$ then $\frac{(A \cap K)+N+L}{N} = \frac{A}{N}$, hence $(A \cap K) + N + L = A$, but $N \subseteq L$ then $(A \cap K) + L = A$, but $A \cap K \ll_J A$ and $J(\frac{A}{L}) = \frac{A}{L}$, then L = A and hence $\frac{L}{N} = \frac{A}{N}$, then $\frac{A}{N} \cap \frac{K+N}{N} \ll_J \frac{A}{N}$.

Proposition (3.10) : Let U,V be a submodules of an R–module M and let V be a J–supplement of U in M if $K \ll_J M$ then V is J–supplement of U + K .

Proof : Let V + (U + K) = M , to prove $V \cap (U + K) \ll_J V$, let $V \cap (U + K) + X = V$, with $J(\frac{V}{X}) = \frac{V}{X}$ to prove V = X , $M = V + (U + K) = V \cap (U + K) + X + (U + K) = X + (U + K) = (U + X) + K$, to prove $J(\frac{M}{U+X}) = \frac{M}{U+X}$, since $\frac{M}{U+X} = \frac{V+(U+K)+X}{U+X} = \frac{V+(U+X)}{(U+X)} \cong \frac{V}{V \cap (U+X)} = \frac{V}{X+(U \cap V)}$ by(Second isomorphism and modular law) . Since $J(\frac{V}{X}) = \frac{V}{X}$, by Corollary (2.3) , we get $J(\frac{V}{X+(U \cap V)}) = \frac{V}{X+(U \cap V)}$, hence $J(\frac{M}{U+X}) = \frac{M}{U+X}$, since $K \ll_J M$ then M = U + X , but M = U + V , and $X \subseteq V$ and $J(\frac{V}{X}) = \frac{V}{X}$, then V = X , by Remark (3.2) .

Proposition (3.11): Let M be any R–module and let V be J–supplement of W in M and $K \subseteq V$ then $K \ll_J M$ if and only if $K \ll_J V$.

Proof: $\Longrightarrow$ Let K + X = V with $J(\frac{V}{X}) = \frac{V}{X}$ to prove V = X , but V + W = M and $V \cap W \ll_J V$, then M=(K + X) + W hence M=K+(X + W) to prove $J(\frac{M}{X+W}) = \frac{M}{X+W}$,since $\frac{M}{X+W} = \frac{V+(X+W)}{(X+W)} \cong \frac{V}{V \cap (X+W)} = \frac{V}{X+(V \cap W)}$ by(Second isomorphism and modular law) . Since $J(\frac{V}{X}) = \frac{V}{X}$ by Corollary (2.3) , we get $J(\frac{V}{X+(V \cap W)}) = \frac{V}{X+(V \cap W)}$, hence $J(\frac{M}{X+W}) = \frac{M}{X+W}$, since $K \ll_J M$ then M = X + W , but M = V + W and $X \subseteq V$ and $J(\frac{V}{X}) = \frac{V}{X}$, then by Remark (3.2) , V = X

$\Longleftarrow$ Clearly by Proposition (2.6.(4)) .

Proposition (3.12) : Let M by any R–module and let V be a J–supplement of U in M , K and T be submodules of V .Then T is J–supplement of K in V if and only if T is J–supplement of U + K in M .

Proof: $\Longrightarrow$ Let T is J–supplement of K in V, then V = T + K and $T \cap K \ll_J V$, Let (U + K) + L = M for $L \subseteq T$ with $J(\frac{T}{L}) = \frac{T}{L}$, to prove T = L . Now $K + L \subseteq V$. Since $\frac{V}{K+L} = \frac{T+(K+L)}{K+L} \cong \frac{T}{T \cap (K+L)} = \frac{T}{L+(K \cap T)}$ by (Second isomorphism and modular law) , and $J(\frac{T}{L}) = \frac{T}{L}$ by Corollary (2.3) , we get $J(\frac{T}{L+(K \cap T)}) = \frac{T}{L+(K \cap T)}$, hence $J(\frac{V}{K+L}) = \frac{V}{K+L}$ and because V is J–supplement of U in M then M = U + V and by Remark (3.2) K + L = V , since $L \subseteq T$ and T is J–supplement of K in V and by Remark (3.2) T = L .

$\Leftarrow$ Let T is J–supplement of U + K in M . Then T + (U + K) = M and T ∩ (U + K) $\ll_J$ T. Let T + K = V to prove T ∩ K $\ll_J$ T since T ∩ K ⊆ T ∩ (U + K) $\ll_J$ T , then by Proposition (2.6.(1)) , T ∩ K $\ll_J$ T , hence T is J–supplement of K in V .

Let U , V be a submodule of a module M , we will say that U and V are **mutual J–supplements** , if U is J–supplement of V in M and V is J–supplement of U in M .

Corollary (3.13) : Let M by any R–module and let U and V be mutual J–supplements in M . L be J–supplement of S in U and T be J–supplement of K in V then L + T is J–supplement of K + S in M .

Proof: Since U = S + L and V is J–supplement of U in M , then by Proposition(3.12) T is J–supplement of S + L + K in M and then (S + L + K) ∩T $\ll_J$ T , since V = K + L and U is J–supplement of V in M , then by Proposition (3.12) , L is J–supplement of S + K + T in M and then (S + K + T) ∩ L $\ll_J$ L , because U = S + L , V = K + T , and M = U + V , then we have M = S + L + K + T = S + K + L + T , then by Proposition (2.6.(2)) , (S + K) ∩ (L + T) ⊆ L ∩ (S + K + T) + T ∩ (S + K + L) $\ll_J$ L + T , hence L + T is J–supplement of K + S in M .

Definition (3.14): Let L and N be a submodules of any R–module M . L is called **weak J–supplement** of N in M . If N + L = M , and N ∩ L $\ll_J$ M , A module M is called **weakly J–supplemented** if every submodule of M has a weak J–supplement in M .

Remarks (3.15) : It is clear that every J–supplemented is weakly J–supplemented . But the converse in general is not true . See the following example . $\mathbb{Q}$ as $\mathbb{Z}$–module is weakly J–supplemented but not J–supplemented .

Proposition (3.16): Let M_1 , K ⊆ M ,and let M_1be a weakly J–supplemented module . If M_1+ K has a weakly J–supplement in M then so does K .

Proof : By assumption there exists N ⊆ M , such that N + (M_1+ K) = M , and N ∩ (M_1+ K) $\ll_J$ M , since M_1 is weakly J–supplemented module there exists L ⊆ M_1 such that (N + K) ∩ M_1+ L = M_1 and (N + K) ∩ L $\ll_J$ M_1 thus K + N + L = M , and (N + K) ∩ L $\ll_J$ M_1 , and by Proposition (2.6.(4)) , (N + K) ∩ L $\ll_J$ M that is L is a weakly J–supplement of N + K in M , we will show that N + L is a weakly J–supplement of K in M , it is clear that (N + L) + K = M , so it enough to show that (N + L) ∩ K $\ll_J$ M . Since (N + L) ∩ K ⊆ N ∩ (M_1 + K) + (N + K) ∩ L$\ll_J$ M , then (N+L) ∩ K $\ll_J$ M . Therefor N + L is a weakly J–supplement of K in M.

Proposition (3.17) : Let M = M_1+ M_2 if M_1 and M_2 are a weakly J–supplemented then M is a weakly J–supplemented.

Proof: Let N be a submodule of M . Since M_1 + M_2 + N = M, trivially has weakly J–supplement in M . And by Proposition (3.16) , M_2 + N has a weakly J–supplement in M . And by Proposition (3.16) , again thus N has a weakly J–supplement in M . So M is a weakly J–supplemented in M .

Proposition (3.18) : Let M be a weakly J–supplemented module and X ⊆ N ⊆ M if X $\ll_J$ M implies that X $\ll_J$ N , then N is a J–supplement submodule of M .

Proof: Suppose that M is a weakly J–supplemented . So M = N + L , L ⊆ M and N ∩ L$\ll_J$ M . By our assumption we get N ∩ L$\ll_J$ N . Hence N is a J–supplement of L in M .

Proposition (3.19) : Let M be a weakly J–supplemented R–module then for every U,V ⊆ M with M = U + V , there exists a weak J–supplement K of U in M with K ⊆ V .

Proof : Assume U,V ⊆ M with M = U + V . Since M is weakly J–supplemented , U ∩ V has a weak J–supplement T in M . In this case M = U∩V+T and (U ∩ V)∩T $\ll_J$ M . Since M = U+V = (U ∩ V) + T (by modular law) , M = U + (V∩T) . Let K = V∩T. Then M = U+K and U ∩ K = U ∩ V ∩ T $\ll_J$ M . Hence K is a weak J–supplement of U in M with K ⊆ V .

Proposition (3.20) : Let M be an R–module and V is a weak J–supplement of U in M for L ⊆ U then $\frac{V+L}{L}$ is a weak J–supplement of $\frac{U}{L}$ in $\frac{M}{L}$.

Proof: Since V is a weak J–supplement of U in M, Then M = U + V and U ∩ V $\ll_J$ M for L ⊆ U we have U ∩ (V+L) = (U∩V) + L (by modular law) , and $\frac{U}{L} \cap (\frac{V+L}{L}) = \frac{(U\cap V)+L}{L}$, since U∩V $\ll_J$ M , it follows $\frac{(U\cap V)+L}{L} \ll_J \frac{M}{L}$, since $\frac{M}{L} = \frac{U+V}{L} = \frac{U}{L} + \frac{V+L}{L}$ and $\frac{U}{L} \cap (\frac{V+L}{L}) = \frac{(U\cap V)+L}{L} \ll_J \frac{M}{L}$. Therefor $\frac{V+L}{L}$ is a weak J–supplement of $\frac{U}{L}$ in $\frac{M}{L}$.

References

1. Inoue, T. **1983**. Sum of hollow modules, *Osaka J. Math* , : 331-336
2. Keskin, D. **2000**. On Lifting Modules, *Comm. Algebra*: 3427-3440.
3. Wisbauer , R. **1991** . *Foundations of Module and Ring Theory* , Gordon and Breach , Philadelphia
4. Kasch, F. **1982**. *Modules and Rings*, Academic press,London.